MAKİNA DİNAMİĞİ

Kaynaklar:

- “Makina Dinamiği” Yıldız Teknik Üniversitesi Yayını, Prof.Necati Tahralı Prof.Dr.Faris Kaya Y.Doç.Dr.İsmail Yüksek Y.Doç.Dr.Rahmi Güçlü.
- “Mekanik Titreşimler” Ders Notları, Prof.Dr.Özgür Turhan.
- “Mekanik Titreşimler” Birsen Kitabevi Yayınları, Prof.Dr.Tuncer Toprak.
- “Vibration Engineering” West Publishing Co., Andrew D. Dimarogonas.
- “Theory Of Vibration With Application” Chapman &Hall, William T. Thomson.

Genel Bilgiler

- Giriş
- Titreşimlerin Sebepleri
- Titreşimlerin Sonuçları
- Sistemlerin Titreşim Analizi
- Titreşim ve İnsan

Titreşimle İlgili Terimler

- Titreşim nedir?

Bir sistemin denge konumu civarında yapmış olduğu salınım hareketine **titreşim** denir.

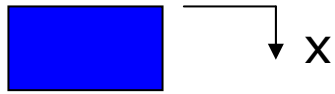
Eğer yapılan salınım hareketi T saniyede kendini tekrar ediyorsa böyle hareketlere **peryodik hareket** denir. En basit peryodik hareket **harmonik hareket** adını alır.

$$x(t) = x(t + nT)$$

x=Yerdeğiştirme	m, rad
t=Zaman	s
T=Peryod	s
n=Peryod sayısı	adet

Titreşim Sistemlerinin Elemanları

- Kütle



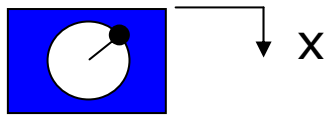
- Yay



- Sönüm



- Kuvvet

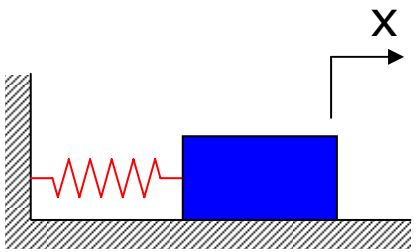


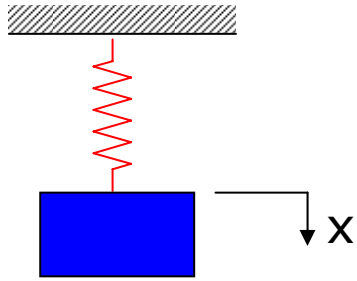
Serbestlik Derecesi

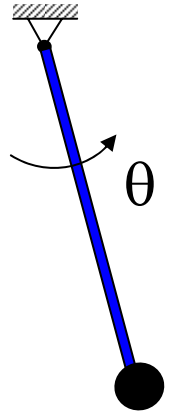
- Hareket halindeki bir sistemin elemanlarının durum ve konumlarını belirleyen parametrelere **koordinat** denir.

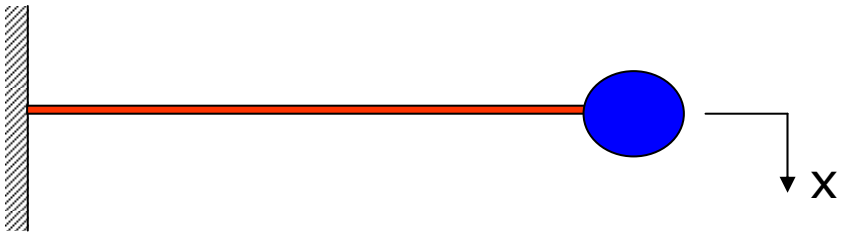
- Bir sistemin bütün parçalarının her hangi bir zamanda konumlarının tamamen belirli olması için gerekli birbirinden bağımsız minimum koordinat **sayısına serbestlik** derecesi denir.

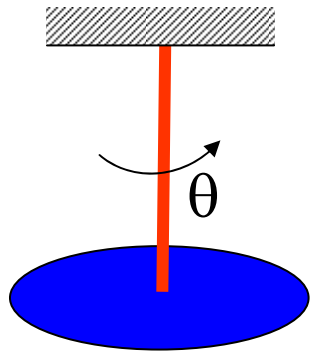
Tek Serbestlik Dereceli Sistemler



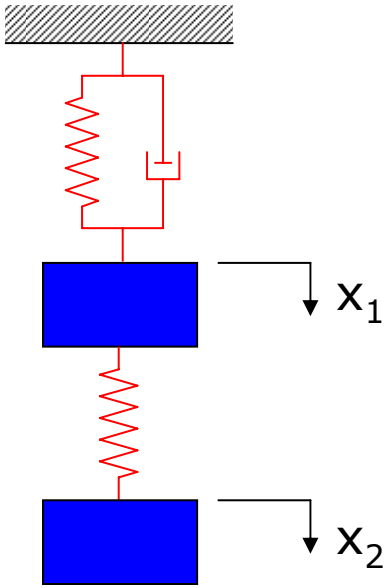


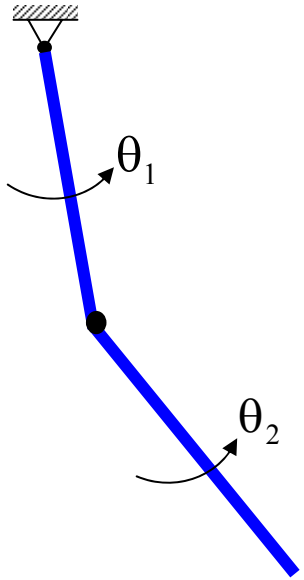


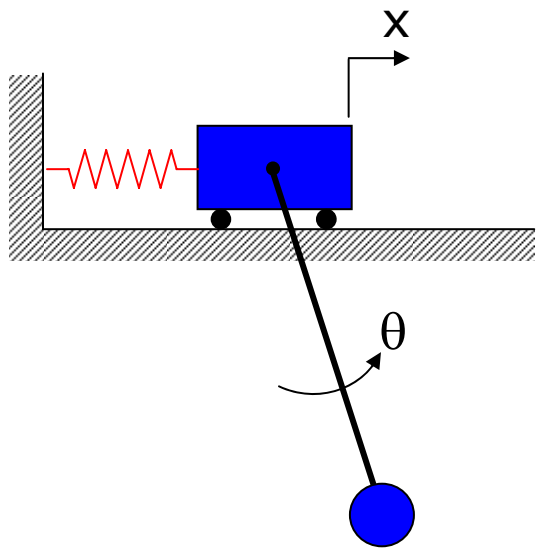


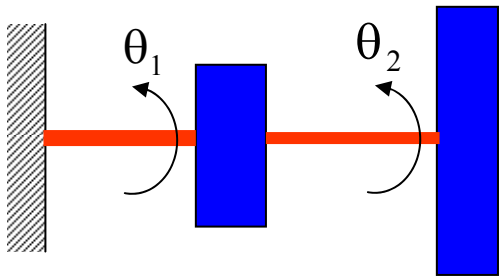


İki Serbestlik Dereceli Sistemler



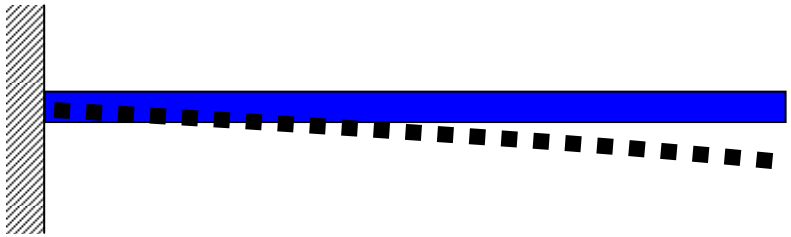






Ayrık ve Sürekli Sistemler

- Sonlu sayıda serbestlik dereceli sistemlere **ayrık sistem** denir.
- Serbestlik derecesi sonsuz olan sistemlere **sürekli sistem** denir.



Sürekli sistem

SI Birim Sistemi

İsim	Birim	Sembol
Uzunluk	Metre	m
Kütle	Kilogram	kg
Zaman	Saniye	s
Kuvvet	Newton	N ($kg.m/s^2$)
Gerilme	Pascal	Pa (N/m^2)
İş	Joule	J (N.m)
Güç	Watt	W (J/s)
Frekans	Hertz	Hz (1/s)
Moment	M	N.m
Kütlesel Atalet Momenti	J	$kg.m^2$
Kesit Atalet Momenti	I	m^4

Harmonik Hareket

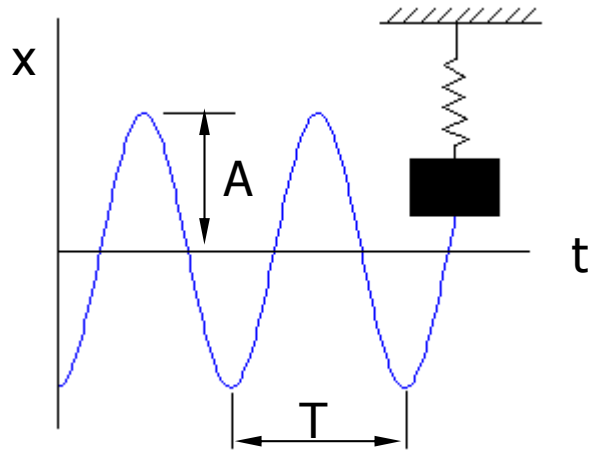
x=Yerdeğiştirme (m,rad)

A=Genlik (m,rad)

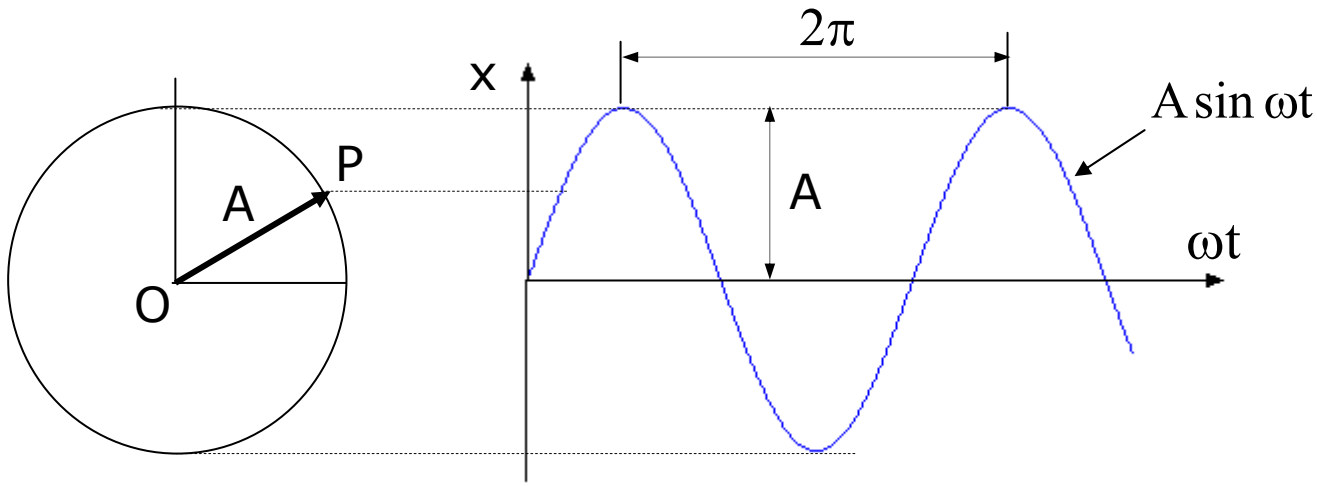
t=Zaman (s)

T=Peryot (s)

$$x = A \sin 2\pi \frac{t}{T}$$



Daire Üzerinde Hareketli Bir Noktanın Harmonik Gösterimi



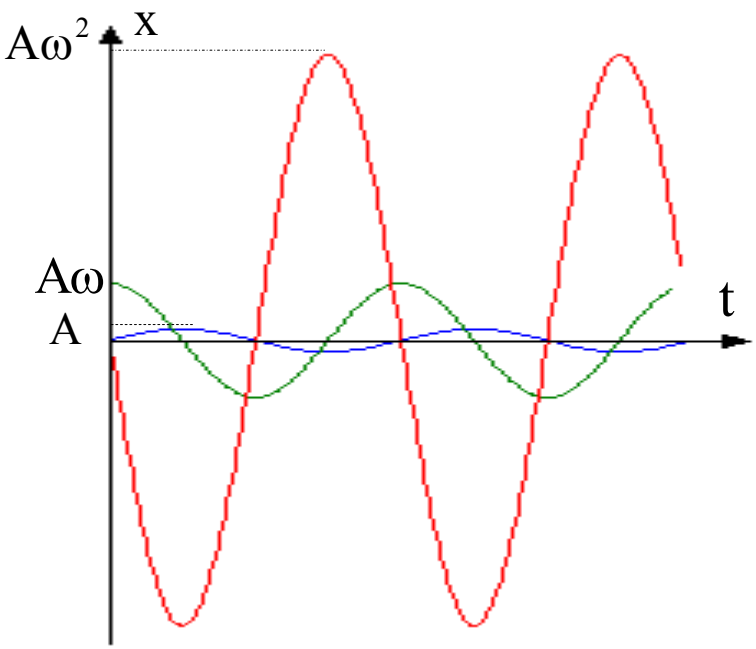
$$\omega = \frac{2\pi}{T} = 2\pi f$$

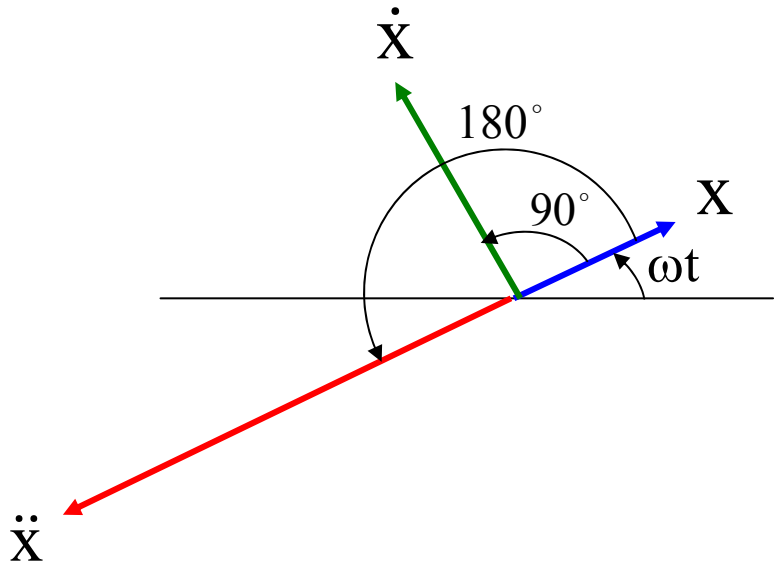
$$x = A \sin \omega t$$

$$\dot{x} = \omega A \cos \omega t = \omega A \sin(\omega t - \pi/2)$$

$$\ddot{x} = -\omega^2 A \sin \omega t = \omega^2 A \sin(\omega t + \pi)$$

Harmonik Harekette Yerdeğiştirme Hız ve İvme Vektörlerinin Gösterimi





Euler Denklemi Yardımı ile Döner Bir Vektörün Gösterimi

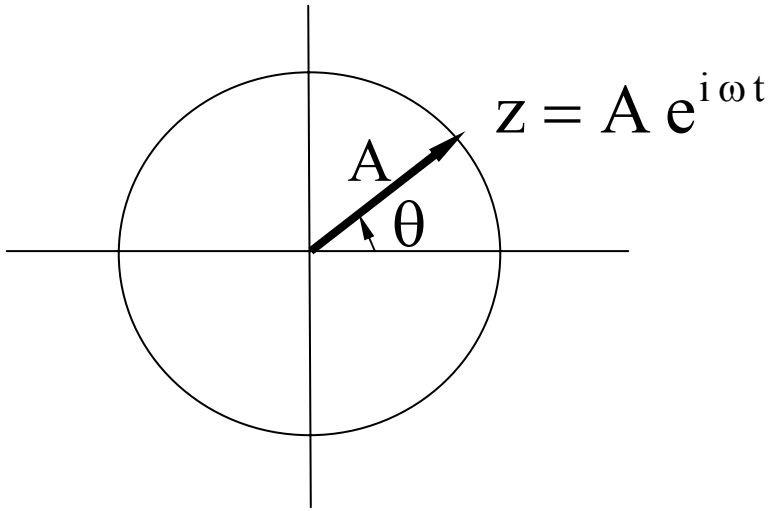
Euler Denklemi $e^{i\theta} = \cos\theta + i\sin\theta$

$z = A e^{i\omega t} = A e^{i\theta}$ $\theta = \omega t$ $A = \sqrt{x^2 + y^2}$

$z = A\cos\omega t + i A\sin\omega t$

$\theta = \tan^{-1}\frac{y}{x}$

$z = x + iy$



Frekansları Aynı Vektörlerin Toplanması

$$x(t) = X_1 \cos(\omega t - \varphi_1) + X_2 \cos(\omega t - \varphi_2) + \ldots\ldots + X_n \cos(\omega t - \varphi_n)$$

$$x(t) = \sum_{i=1}^{n} X_i \cos(\omega t - \varphi_i)$$

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$ bağıntısı kullanılarak

$$x(t) = \sum_{i=1}^{n} X_i (\cos\omega t \cos\varphi_i + \sin\omega t \sin\varphi_i)$$

$$= \left(\sum_{i=1}^{n} X_i \cos\varphi_i \right) \cos\omega t + \left(\sum_{i=1}^{n} X_i \sin\varphi_i \right) \sin\omega t$$

Buradan,

$$A = \sum_{i=1}^{n} X_i \cos\varphi_i \qquad B = \sum_{i=1}^{n} X_i \sin\varphi_i$$ yazılırsa,

$$x(t) = A\cos\omega t + B\sin\omega t$$ bulunur.

$$X = \sqrt{A^2 + B^2} = \sqrt{\left(\sum_{i=1}^{n} X_i \cos\varphi_i\right)^2 + \left(\sum_{i=1}^{n} X_i \sin\varphi_i\right)^2}$$

$$\varphi = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{\sum_{i=1}^{n} X_i \sin\varphi_i}{\sum_{i=1}^{n} X_i \cos\varphi_i}$$

$$x(t) = X\cos(\omega t - \varphi)$$

Problem:

$$F_1(t) = 250\, e^{i(20\pi t + \pi/3)}$$

$$F_2(t) = (500 + i\,288.67)\, e^{i\,20\pi t}$$

$$F_3(t) = 500\,(\cos 20\pi\, t + i \sin 20\pi\, t)$$

$$F(t) = ?$$

Toplam vektörü analitik ve nümerik olarak hesaplayın ve 2 peryod için grafiğini çiziniz.

Çözüm:

$$F_1(t) = 250\, e^{i(20\pi t+\pi/3)}$$

$$F_2(t) = (500 + i\,288.67)\, e^{i\,20\pi t} = 577.3\, e^{i(20\pi t+30^\circ)}$$

$$F_3(t) = 500\,(\cos 20\pi\, t + i \sin 20\pi\, t) = 500\, e^{i\,20\pi t}$$

$$\sum F = F_1 + F_2 + F_3 = 250\, e^{i(20\pi t+\pi/3)} + 577.3\, e^{i(20\pi t+30^\circ)} + 500\, e^{i\,20\pi t}$$

$$\sum F = e^{i\,20\pi t}\left(250\, e^{i\pi/3} + 577.3\, e^{i\pi/6} + 500\right)$$

$$\sum F = (1125 + i\,505.17)\, e^{i20\pi t}$$

$$\sum F = 1233.2\, e^{i(20\pi t+24.18^\circ)}$$

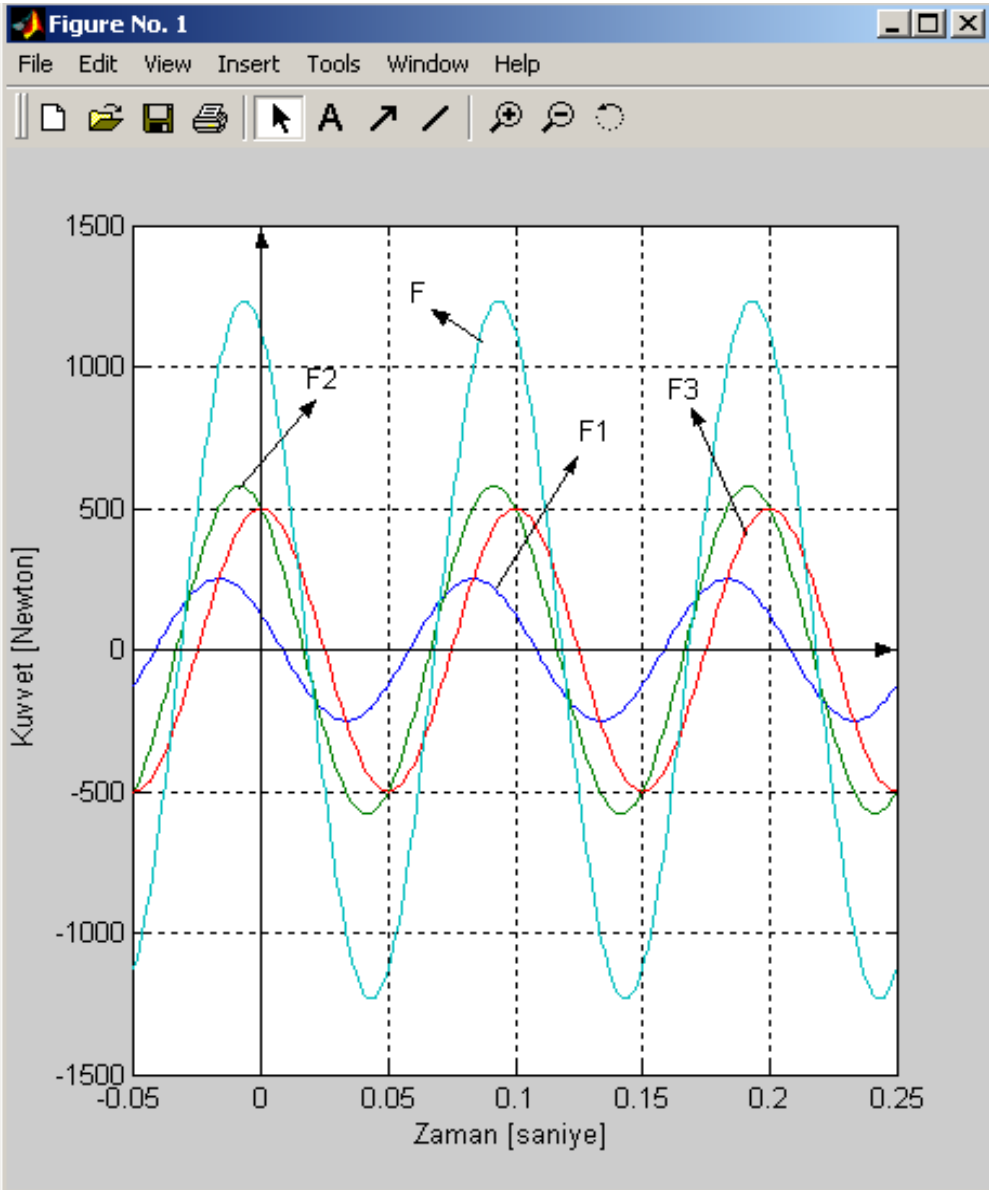
Problemin MATLAB programı ve grafik gösterimi

C:\MATLAB6p5\work\Untitled7.m*

File Edit View Text Debug Breakpoints Web Window Help

```
%*** 3 Titreşim Vektörünü Toplayan Program ***
clear
t=-0.05:0.001:0.25;
F1=250*exp(i*(20*pi*t+pi/3));
F2=(500+i*288.67)*exp(i*20*pi*t);
F3=500*(cos(20*pi*t)+i*sin(20*pi*t));
F=F1+F2+F3;
plot(t,F1,t,F2,t,F3,t,F)
grid
axis([-0.05 0.25 -1500 1500])
xlabel('Zaman [saniye]')
ylabel('Kuvvet [Newton]')
```

script Ln 4 Col 1



Frekansları Farklı Vektörlerin Toplanması

$$x(t) = X_1 \cos(\omega_1 t - \varphi_1) + X_2 \cos(\omega_2 t - \varphi_2) + \ldots\ldots + X_n \cos(\omega_n t - \varphi_n)$$

$$x(t) = \sum_{i=1}^{n} X_i \cos(\omega_i t - \varphi_i)$$

Toplam vektörün peryodik olması için, $T = k_i T_i = k_i \dfrac{2\pi}{\omega_i}$ olmalıdır.

Bu ise ancak T'nin T_i'lerin tam katı olması ile mümkündür. Buna göre, T T_i'lerin ortak katlarının en küçüğü olacaktır.

ÖRNEK:

$$x_1(t) = 50\cos(2\pi t)$$
$$x_2(t) = 175\cos(0.8\pi t)$$
$$x_3(t) = 30\sin(8.78\pi t)$$
$$x_4(t) = 225\cos(\pi t/7)$$
$$x(t) = ?$$

Yukarıdaki titreşim vektörlerini sayısal olarak toplamını bulunuz. Toplam vektörün peryodik olduğunu gösteriniz.

Frekansları Birbirine Yakın Vektörlerin Toplanması

$$\omega_1 \approx \omega_2$$

$$x(t) = X_1 \cos(\omega_1 t - \varphi_1) + X_2 \cos(\omega_2 t - \varphi_2)$$

Bu ifadeyi yeniden düzenleyelim,

$$x(t) = \frac{X_1 + X_2}{2}\left[\cos(\omega_1 t - \varphi_1) + \cos(\omega_2 t - \varphi_2)\right] + \frac{X_1 - X_2}{2}\left[\cos(\omega_1 t - \varphi_1) - \cos(\omega_2 t - \varphi_2)\right]$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \quad \text{ve} \quad \cos\alpha - \cos\beta = -2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$$

Trigonometrik bağıntılarından yararlanarak,

$$\tilde{A}(t) = (X_1 + X_2)\cos\frac{(\omega_1 + \omega_2)t - (\varphi_1 - \varphi_2)}{2}$$

ve

$$\tilde{B}(t) = -(X_1 - X_2)\sin\frac{(\omega_1 + \omega_2)t - (\varphi_1 - \varphi_2)}{2}$$

Tanımları altında,

$$x(t) = \tilde{A}(t)\cos\left(\frac{\omega_1+\omega_2}{2}t - \frac{\varphi_1+\varphi_2}{2}\right) + \tilde{B}(t)\sin\left(\frac{\omega_1+\omega_2}{2}t - \frac{\varphi_1+\varphi_2}{2}\right)$$

Yazılabilir. Burada,

$$\tilde{X}(t) = \sqrt{\tilde{A}(t) + \tilde{B}(t)} \qquad \text{ve} \qquad \tilde{\varphi}(t) = \frac{\varphi_1+\varphi_2}{2} + \tan^{-1}\frac{\tilde{B}(t)}{\tilde{A}(t)}$$

$$\tilde{X}(t) = \sqrt{X_1^2 + X_2^2 + 2X_1X_2\cos\left[(\omega_1-\omega_2)t - (\varphi_1-\varphi_2)\right]}$$

$$\tilde{\varphi}(t) = \frac{\varphi_1+\varphi_2}{2} + \tan^{-1}\left[\frac{X_1-X_2}{X_1+X_2}\tan\left(\frac{\omega_1-\omega_2}{2}t - \frac{\varphi_1-\varphi_2}{2}\right)\right]$$

ifadeleri kullanılarak,

$$x(t) = \tilde{X}(t)\cos\left(\frac{\omega_1+\omega_2}{2}t - \tilde{\varphi}(t)\right)$$ toplam ifadesi bulunur.

ÖRNEK:

$$F_1(t) = 2500\, e^{i\,10.5\,t}$$

$$F_2(t) = 500\,(\cos 10t + i \sin 10t)$$

$$F(t) = ?$$

Verilen kuvvetleri analitik ve sayısal olarak toplayınız ve t=20 s için grafiğini çiziniz.

Yay Elemanları

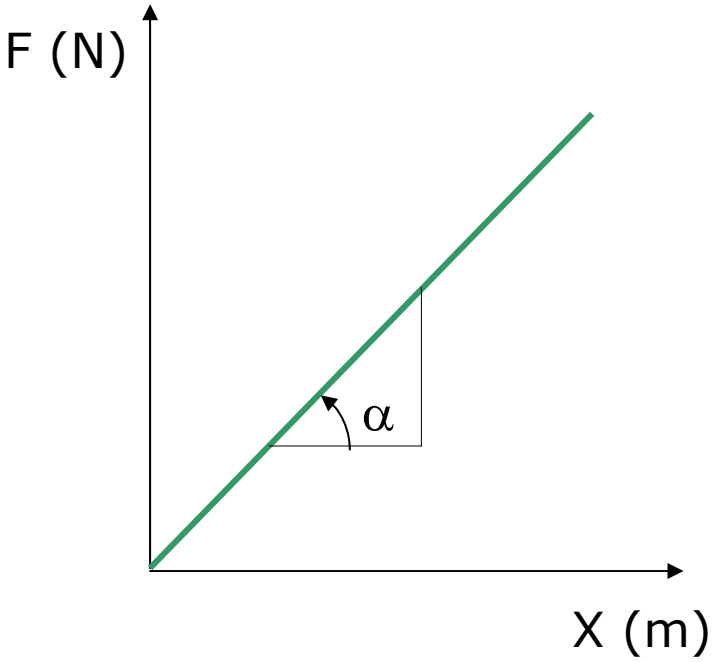
Helisel Yaylar



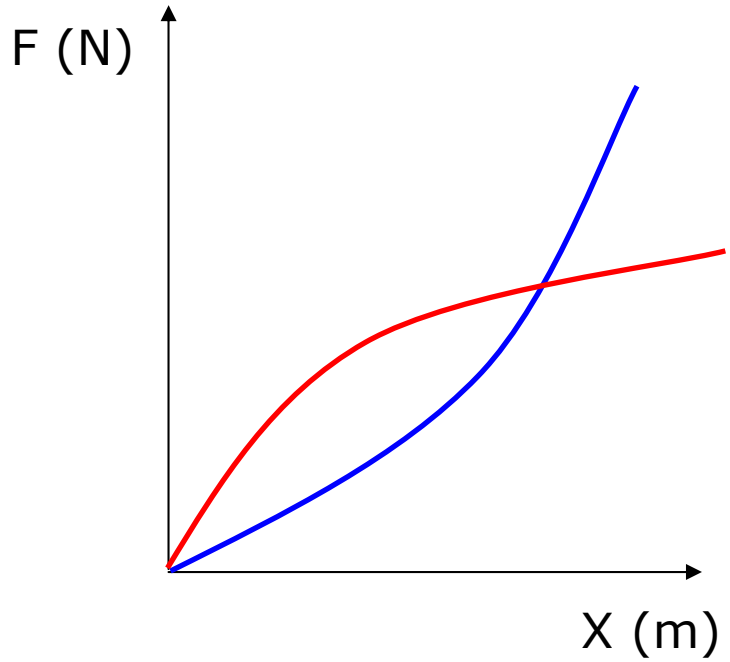
Yaprak Yaylar



Yay Karakteristikleri

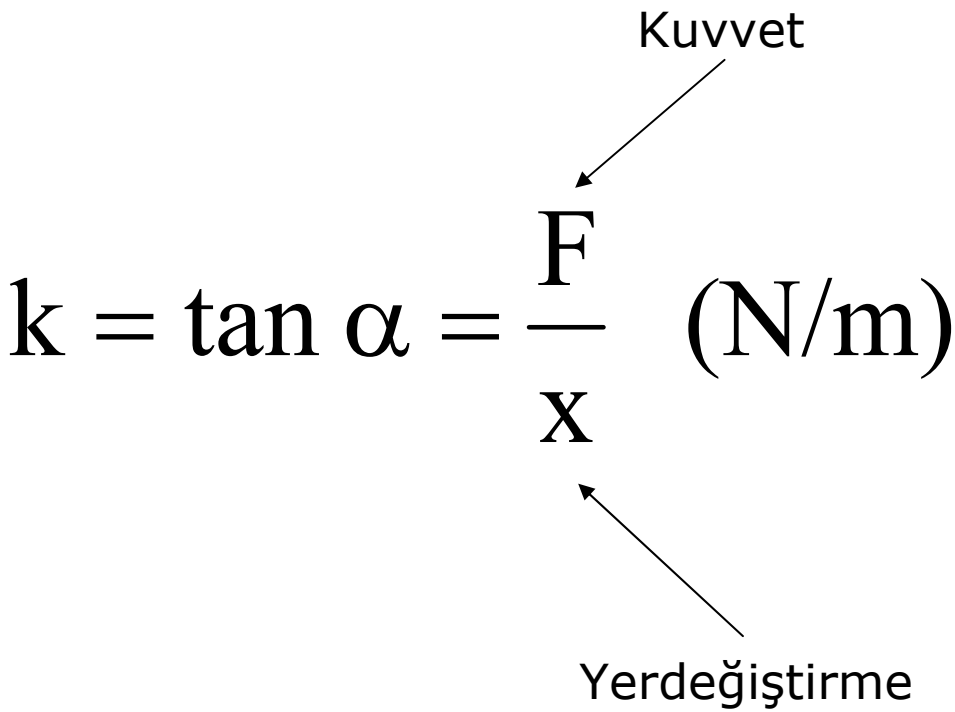


Lineer (doğrusal) yay karakteristiği

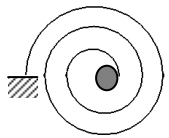


Non-Lineer (doğrusal olmayan) yay karakteristiği

Yay Katsayısı



Yay Katsayısı Tablosu



$$k = \frac{E\,I}{L}$$



$$k = \frac{E\,A}{L}$$



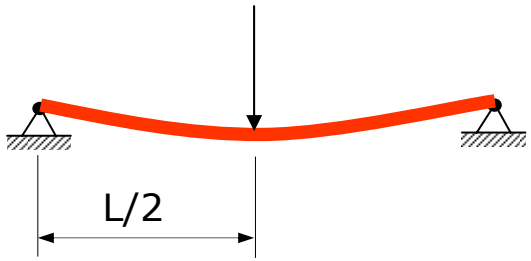
$$k = \frac{G\,I_p}{L}$$



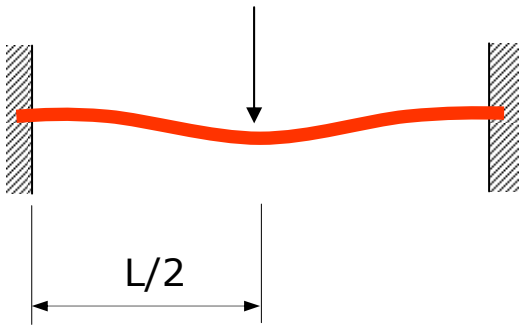
$$k = \frac{G\,d^4}{64\,n\,R^3}$$



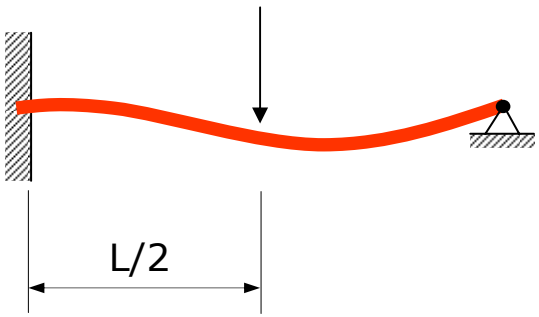
$$k = \frac{3EI}{L^3}$$



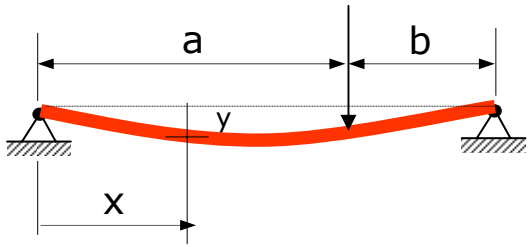
$$k = \frac{48\,E\,I}{L^3}$$



$$k = \frac{192\,E\,I}{L^3}$$

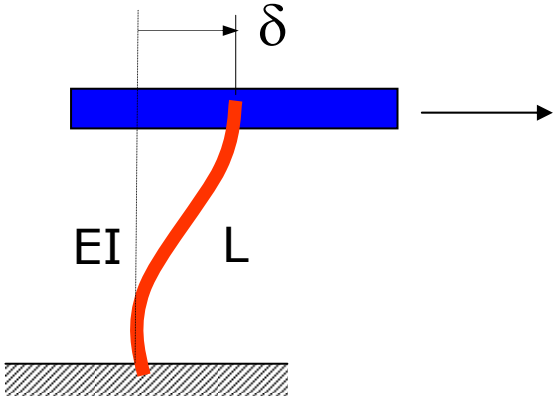


$$k = \frac{768\,E\,I}{7\,L^3}$$

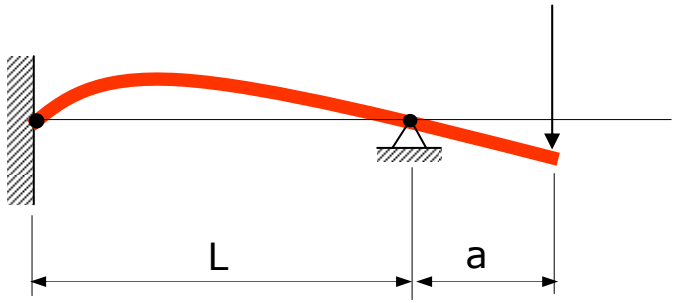


$$k = \frac{3EIL}{a^2b^2}$$

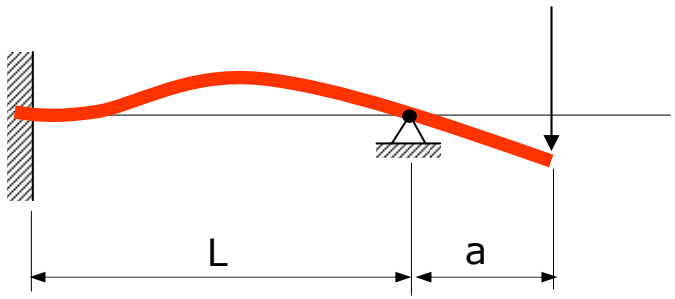
$$y_x = \frac{Pbx}{6EIL}\left(L^2 - x^2 - b^2\right)$$



$$k = \frac{12EI}{L^3}$$

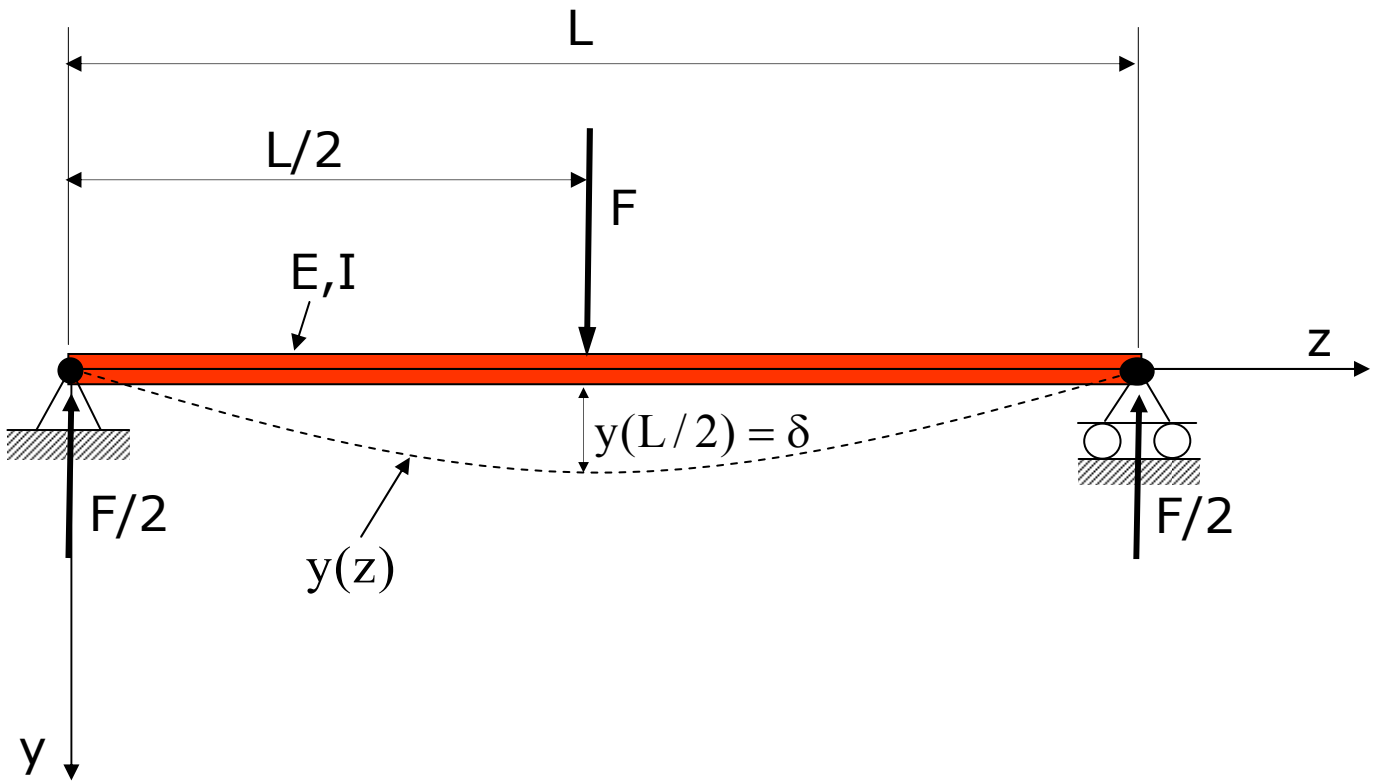


$$k = \frac{3EI}{(L+a)a^2}$$



$$k = \frac{24EI}{a^2(3L+8a)}$$

Problem:Şekildeki tam ortasından yüklü basit mesnetli kirişin eşdeğer yay katsayısını hesaplayınız.



Çözüm:

Orta noktasına etkiyen bir kuvvetin altında, kirişte meydana gelen elastik çökmeleri veren elastik eğri denklemi aşağıdaki diferansiyel denklem ile tarif edilir.

$$E\,I\frac{d^2y}{dz^2} = -M_e = -\frac{F}{2}z \qquad 0 \le z \le \frac{L}{2}$$

Mesnet ve yükleme durumu göz önüne alınarak,aşağıdaki sınır şartları yazılabilir.

$$y(0) = 0 \quad , \quad \frac{dy(L/2)}{dz} = 0$$

E:Elastite modülü

I:Kesit atalet momenti

M_e=Kirişin z kesitindeki eğilme momenti

Elastik eğri denklemi çözülürse,

$$E\,I\,y(z) = -\frac{F}{12}z^3 + c_1 z + c_2$$

Burda, c_1 ve c_2 sabitleri sınır şartlarından hesaplanır.

$$c_1 = \frac{F\,L^2}{16} \quad , \quad c_2 = 0$$

Bulunan sabitler yukarıda yerlerine konursa,

$$y(z) = \frac{F\,L^3}{48\,E\,I}\left(3\frac{z}{L} - 4\frac{z^3}{L^3}\right) \qquad 0 \le z \le \frac{L}{2}$$

olarak bulunur.

Kuvvetin etkidiği noktadaki yer değiştirme,

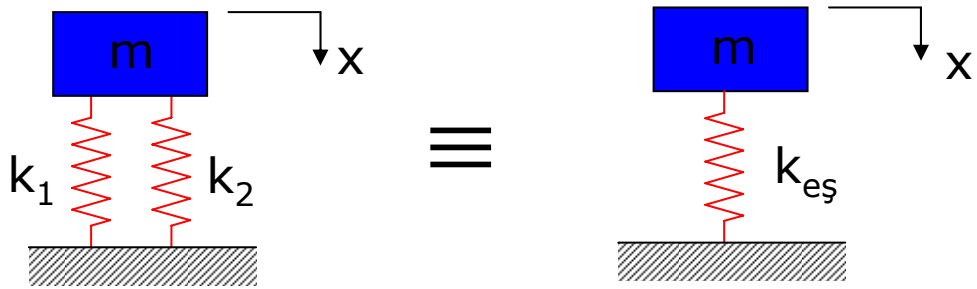
$$\delta = y\left(\frac{L}{2}\right) = \frac{F\,L^3}{48\,E\,I}$$

dir. Buradan,

$$k_{eş} = \frac{F}{\delta} = \frac{48\,E\,I}{L^3}$$

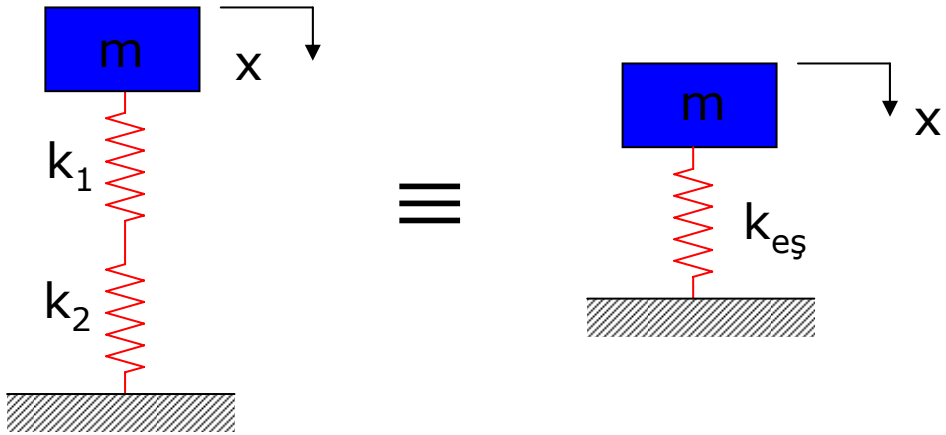
Bulunur.

Yayların Paralel Bağlanması



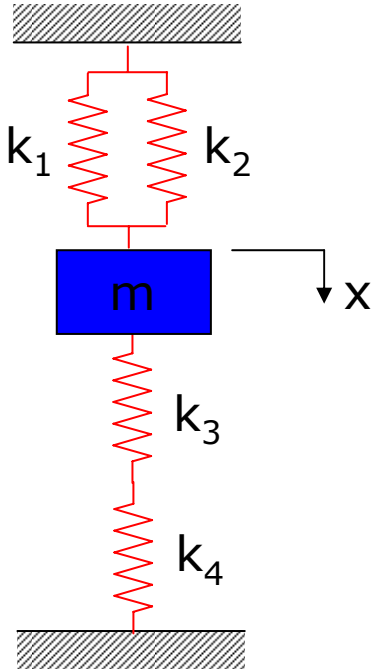
$$k_{eş} = k_1 + k_2 + k_3 + ... + k_n = \sum_{i=1}^{n} k_n$$

Yayların Seri Bağlanması



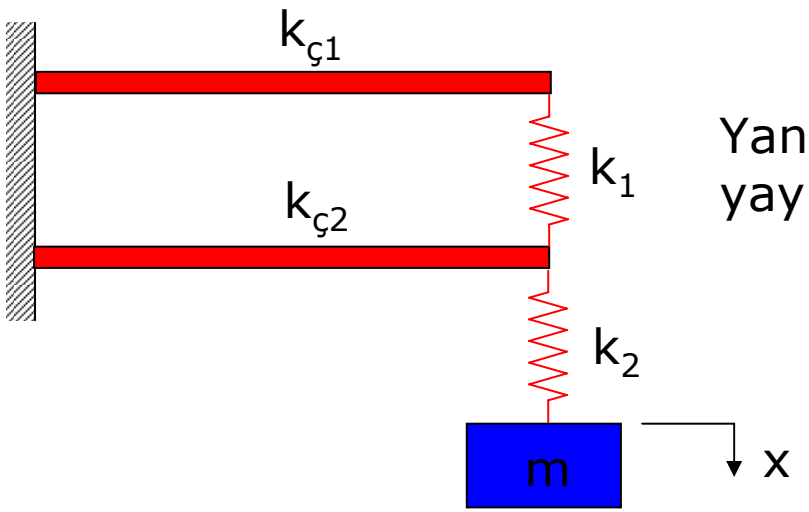
$$\frac{1}{k_{eş}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + ... + \frac{1}{k_n} = \sum_{i=1}^{n} \frac{1}{k_i}$$

ÖRNEK:



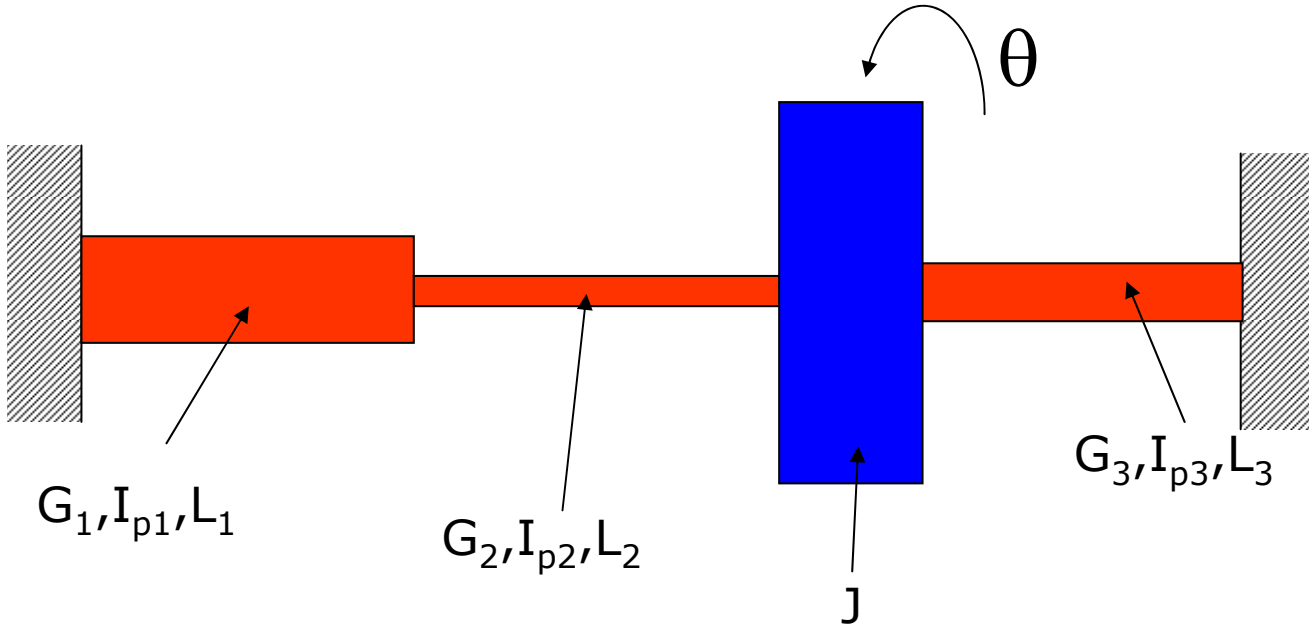
Yandaki sistemin eşdeğer yay katsayısını hesaplayınız.

ÖRNEK:



Yandaki sistemin eşdeğer yay katsayısını hesaplayınız.

ÖRNEK:



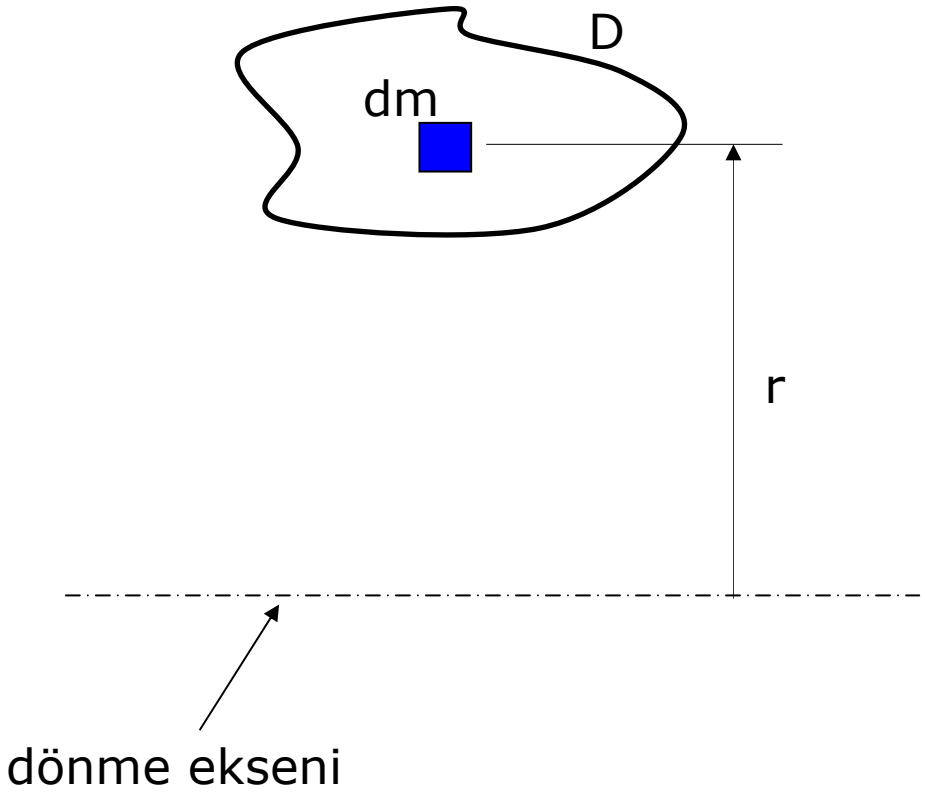
Yukarıdaki burulma sistemin eşdeğer yay katsayısını hesaplayınız.

Sönüm Elemanları

- Viskoz sönüm
- Coulumb (kuru sürtünme) sönümü
- Malzeme (histeresiz) sönüm
- Sıkıştırılmış yağ (squeeze-film) damperi
- Elekro-manyetik damper
- Elektro-viskoz damper
- Piezo-elektrik damper

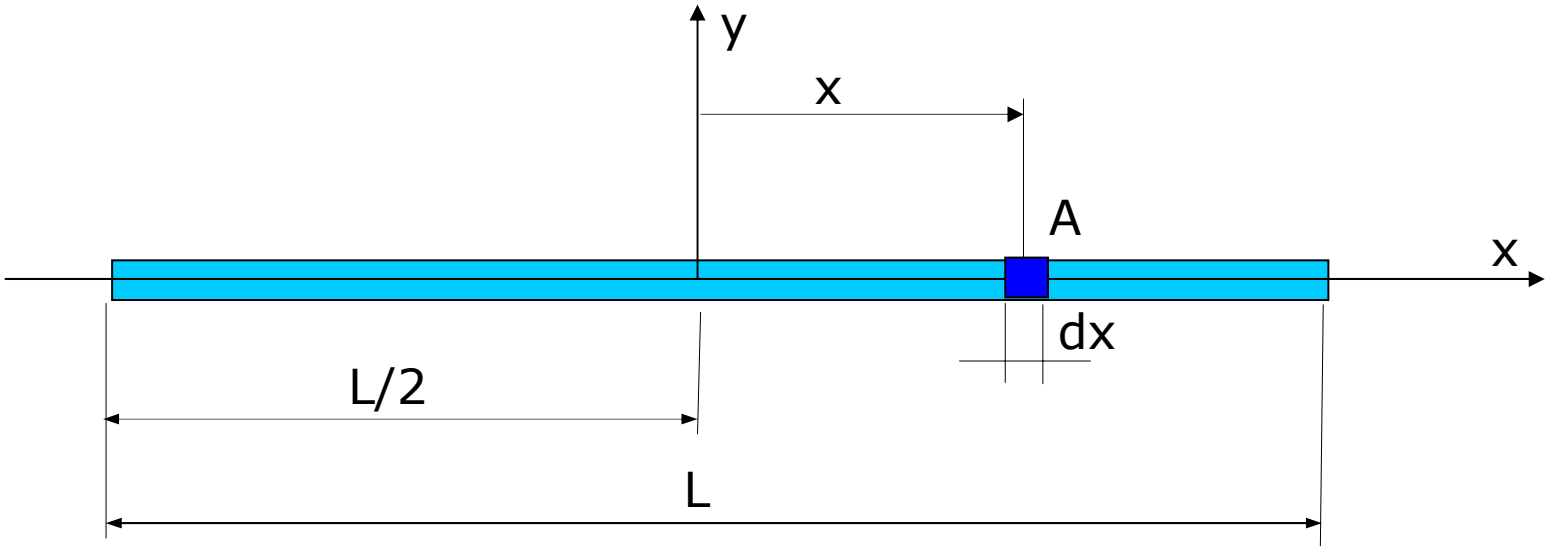
Kütle ve Atalet Elemanları

Bir cismin bir dönme eksenine göre kütlesel atalet momentinin tanımı:



$$J = \int_D r^2 dm$$

Problem: Orta noktasından mafsallı ve sabit kesitli bir çubuğun kütlesel atalet momentinin bulunması



Çözüm:

Elemanter hacim $dV = A\,dx$

Elemanter kütle $dm = \rho\,dV$

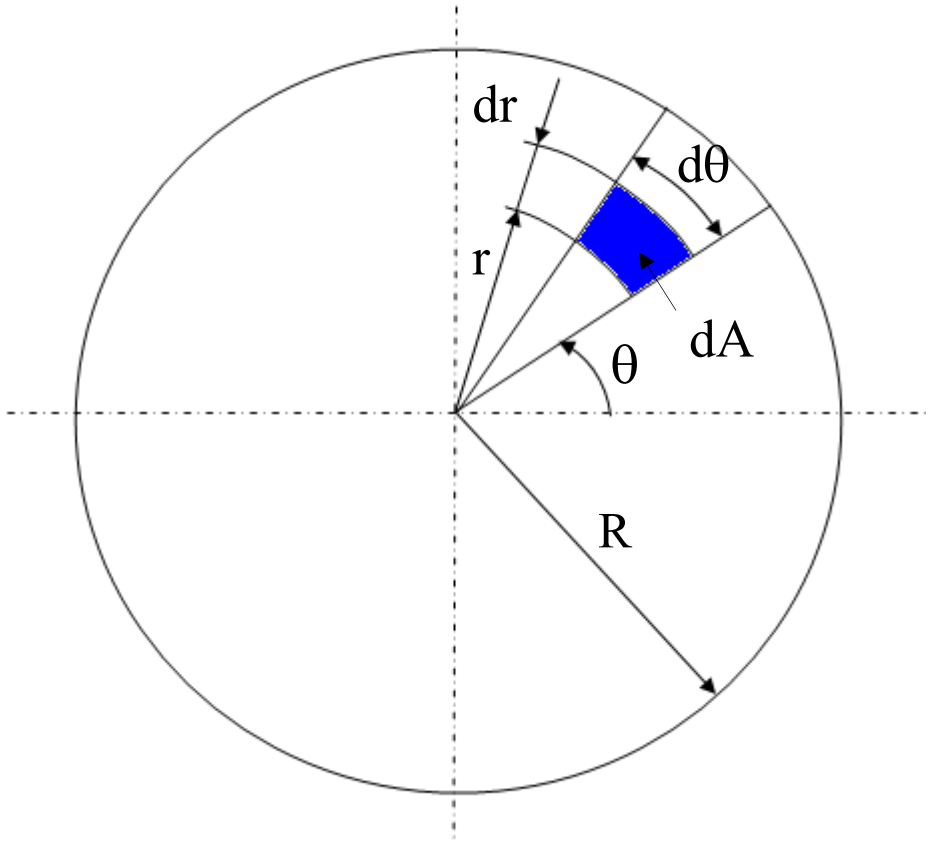
Kütlesel atalet momentinin tanımından $J = \int_D r^2 dm$

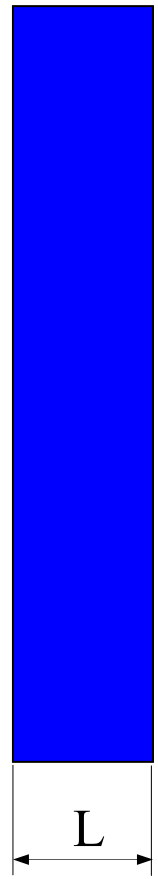
$$J = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho\, A\, x^2 dx = \rho\, A \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 dx$$

$$J = \rho\, A \left| \frac{x^3}{3} \right|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{1}{12} \rho\, A\, L^3$$ bulunur.

burada, $m = \rho\, A\, L \quad \Rightarrow \quad J = \frac{1}{12} m\, L^2$

Problem: Bir diskin dönme eksenine göre kütlesel atalet momentinin bulunması.





Çözüm:

Elemanter alan $dA = r.\sin d\theta.dr$

Elemanter hacim $dV = L.dA = L.\, r.\, \sin d\theta.dr$

Elemanter kütle $dm = \rho.dV = \rho.L.r.\sin d\theta.dr$

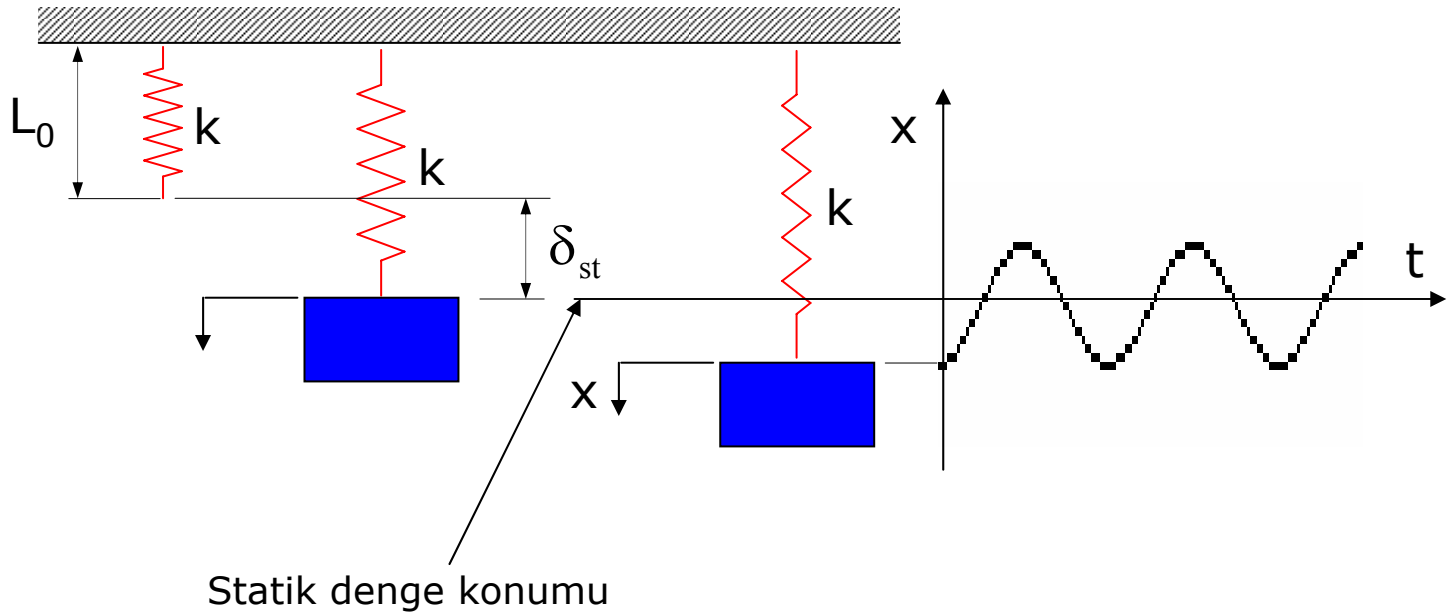
$\sin d\theta \cong d\theta \quad \Rightarrow \quad dm = \rho.Lr.d\theta.dr$ bulunur.

$$J = \oint_D r^2 dm = \int_0^{2\pi}\int_0^{R} \rho.L.r^3.d\theta.dr = \frac{1}{2}\rho.L.\pi.R^4$$

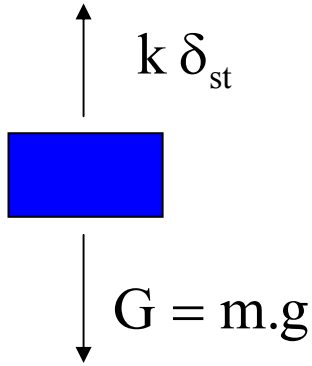
$$m = \rho.V = \rho.\pi.R^2.L \quad \Rightarrow \quad J = \frac{1}{2}m.R^2$$

Sönümsüz Serbest Titreşim

Düşey konumda kütle-yay sisteminin hareket denklemi:



Serbest cisim diyagramı



Statik denge konumu: $\Sigma F_y = 0$

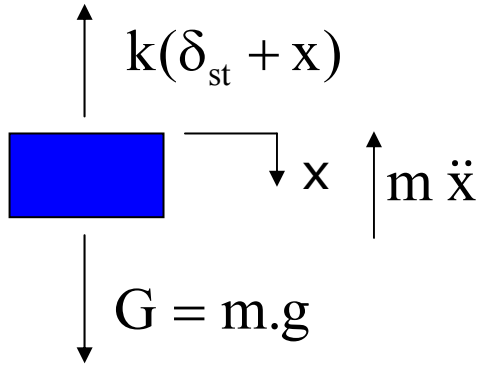
$$G = m.g = k.\delta_{st}$$

$$\frac{k}{m} = \frac{g}{\delta_{st}} = \omega_n^2$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$$

Burada, ω_n sistemin tabii frekansıdır.

Newton'un 2. kanunu uygularsak,

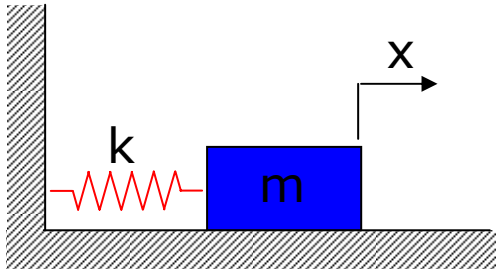


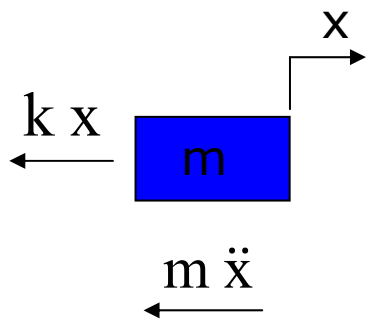
$$\Sigma F = m.a \quad \Rightarrow \quad m\ddot{x} = -k\left(\delta_{st} + x\right) + G$$

$G = m.g = k.\delta_{st}$ olduğundan,

$m\ddot{x} + k\,x = 0$ bulunur.

Yatay konumda kütle-yay sisteminin hareket denklemi:





Newton'un 2. kanunu uygularsak,

$$\Sigma F = m.a$$

$$m\,\ddot{x} = -k\,x \quad \Rightarrow \quad m\,\ddot{x} + k\,x = 0$$

bulunur.

Hareketin Diferansiyel Denkleminin Enerji Metodu ile Bulunması

Bu metoodun kullanılabilmesi için titreşim sisteminin;

- Sönümsüz
- Tek serbestlik dereceli olması gerekir.

$$E_k + E_p = C = \text{sabit}$$

$$\frac{d}{dt}\left(E_k + E_p\right) = 0$$

Sönümsüz Serbest Titreşim Hareket Denkleminin Bulunması

$$m\ddot{x} + k\,x = 0$$

Bu diferansiyel denklemin çözümünün

$$x = A\,e^{s\,t}$$

biçiminde olduğunu biliyoruz. Burada, A ve s integrasyon sabitleridir. Çözüm kabulünü türetirsek,

$$\dot{x} = s\,A\,e^{s\,t}$$

$$\ddot{x} = s^2\,A\,e^{s\,t}$$

Bulunur. Bunlar yukarıdaki diferansiyel denklemde yerine konursa,

$$\left(m\,s^2 + k\right) A\, e^{st} = 0$$

Burada, $A\,,\ e^{s\,t} \neq 0$ dır.

$$m\,s^2 + k = 0$$

Bulunur, bu denkleme **karakteristik denklem** denir.
Karakteristik denklemin kökleri,

$$s_{1,2} = \mp\sqrt{-\frac{k}{m}} = \mp i\,\omega_n$$

dir.

Bu durumda, hareket denklemi:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{-i\omega_n t} + A_2 e^{i\omega_n t}$$

A_1 ve A_2 başlangıç şartlarından bulunacak katsayılardır.

$e^{\mp i\theta t} = \cos\theta.t \mp i.\sin\theta.t$ eşitliği kullanılırsa,

$$x(t) = A_1(\cos\omega_n t - i\sin\omega_n t) + A_2(\cos\omega_n t + i\sin\omega_n t)$$

$$x(t) = (A_1 + A_2)\cos\omega_n t - i(A_1 - A_2)\sin\omega_n t$$

$B_1 = (A_1 + A_2)$ ve $B_2 = (A_1 - A_2)$ olmak üzere,

$$x(t) = B_1 \cos \omega_n t + B_2 \sin \omega_n t$$ olur.

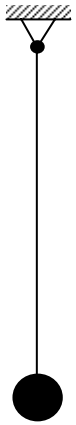
Başlangıç şartları $t = 0 \Rightarrow \begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases}$ olsun.

Bu durumda, $B_1 = x_0$, $B_2 = \dfrac{\dot{x}_0}{\omega_n}$ bulunur.

Buradan başlangıç şartlarına bağlı hareket denklemi aşağıdaki şekilde bulunur.

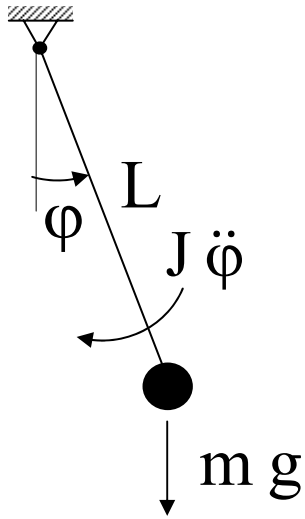
$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

Problem: Aşağıdaki sarkacın diferansiyel denklemini çıkarınız. Tabii frekansını ve periyodunu hesaplayınız.



L uzunluğunda ağırlıksız bir ipin ucuna m kütlesi asılmıştır.

Çözüm:



Newton'un 2. kanunu uygulanırsa,

$$\Sigma M = J_{Top}\ddot{\varphi}$$

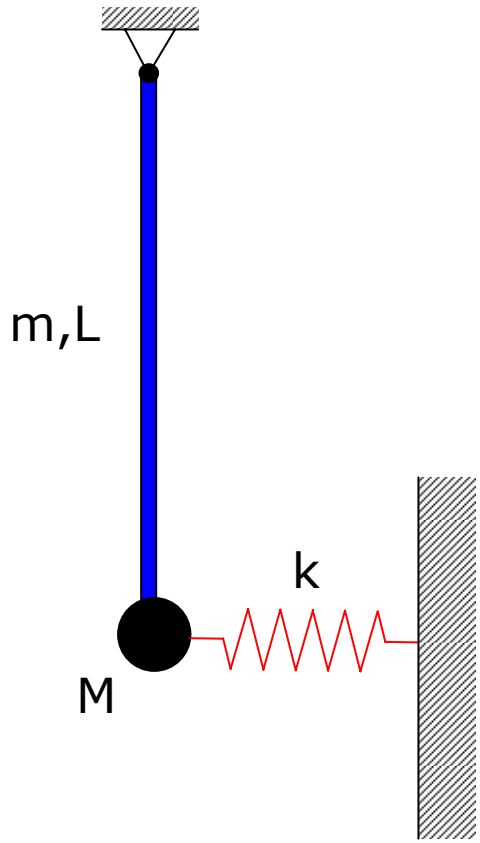
$$J\ddot{\varphi} = -m\,g.L\sin\varphi$$

$$J = m\,L^2 \qquad \varphi << 0 \;\Rightarrow\; \sin\varphi \cong \varphi$$ alınabilir.

$$m\,L^2\ddot{\varphi} + m\,g\,L\,\varphi = 0 \;\Rightarrow\; \ddot{\varphi} + \frac{g}{L}\varphi = 0$$ buradan,

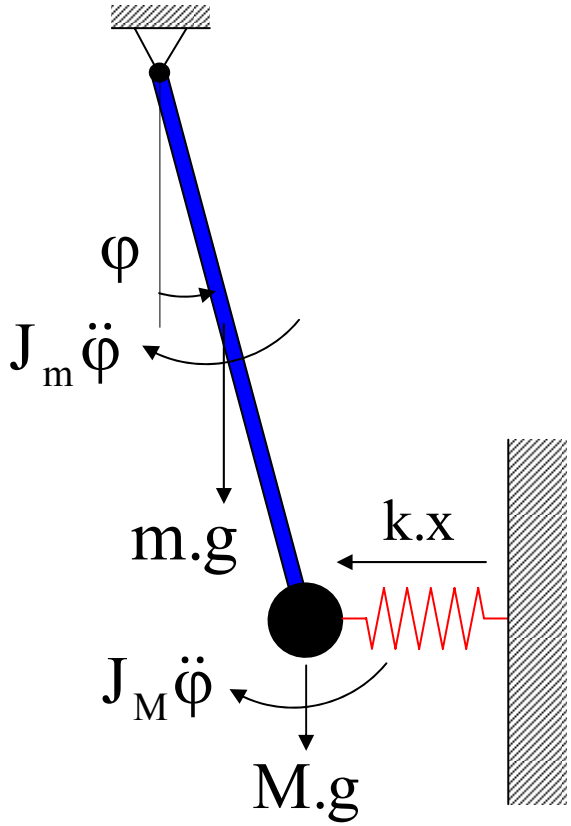
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}}$$ rad/s $$\qquad T = \frac{2\pi}{\omega_n} = \frac{1}{f_n}$$ s

Problem: Aşağıda denge konumunda verilen sistemin diferansiyel denklemini çıkarıp tabii frekansını hesaplayınız.



Çözüm:

Sistemin denge konumunu bir miktar bozalım ve oluşan kuvvetleri gösterelim.



Newton'un 2. kanunu uygulanırsa,

$$\Sigma M = J_{Top}\ddot{\varphi}$$

$$J_m\ddot{\varphi} + J_M\ddot{\varphi} = -m\,g.\frac{L}{2}\sin\varphi - M\,g.L\sin\varphi - k\,x.L\cos\varphi$$

$\varphi << 0 \Rightarrow \sin\varphi \cong \varphi \quad \cos\varphi \cong 1$ alınabilir.

Yaydaki sıkışma miktarı $x = L.\sin\varphi = L\,\varphi$

$$\left(\frac{1}{3}m\,L^2 + M\,L^2\right)\ddot{\varphi} + \left(\frac{1}{2}m\,g.L + M\,g.L + k\,L^2\right)\varphi = 0$$

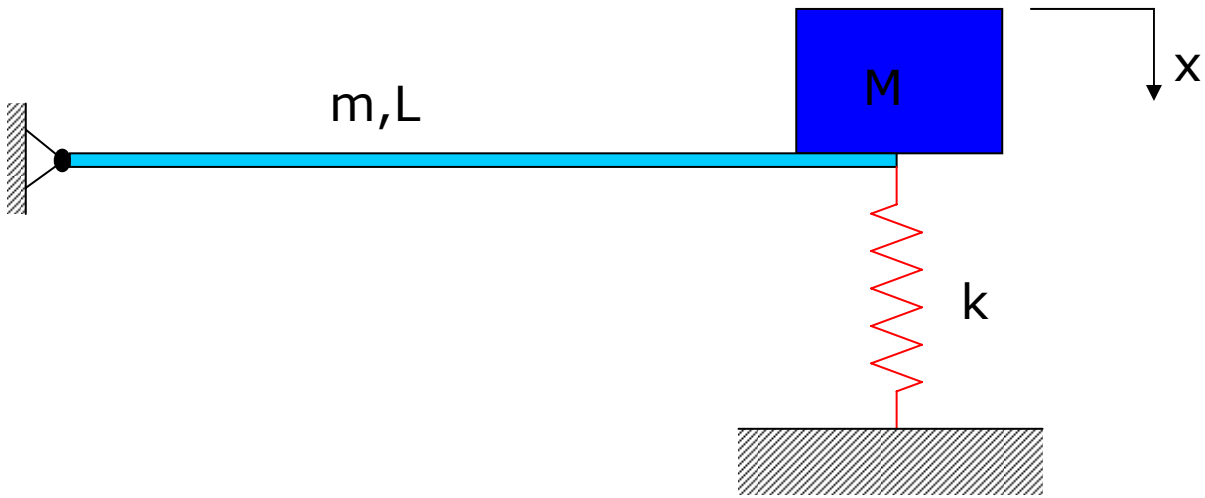
$$J_m = \frac{1}{3} m L^2 , \quad J_M = M L^2$$ dir.

sadeleştirme yapılırsa,

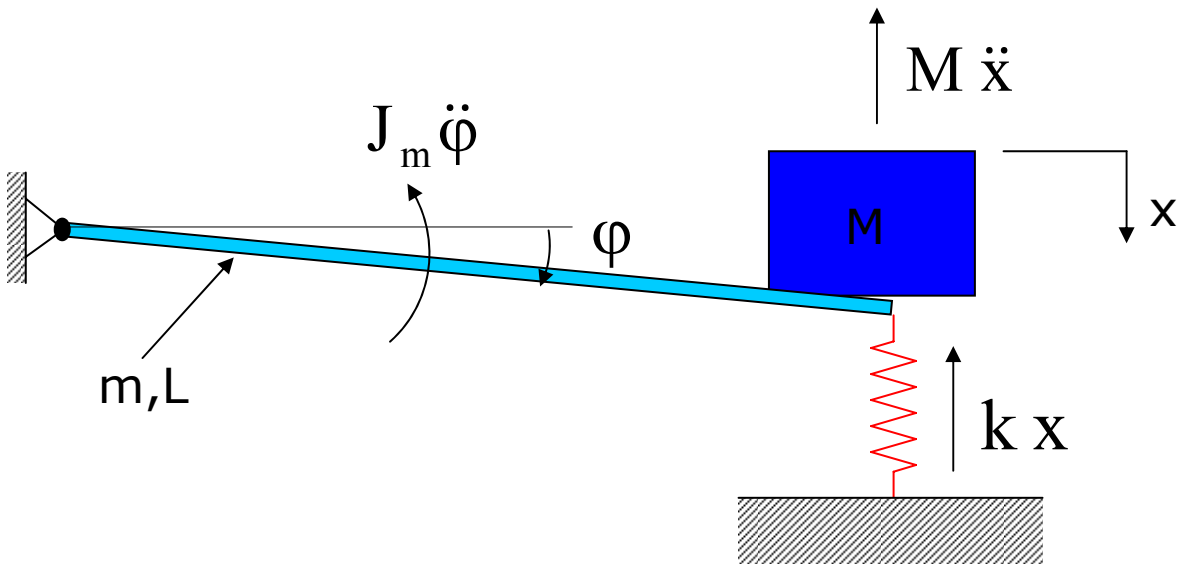
$$\left(\frac{1}{3} m + M\right) L \ddot{\varphi} + \left(\frac{1}{2} m g + M g + k L\right) \varphi = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{\frac{1}{2} m g + M g + k L}{\left(\frac{1}{3} m + M\right) L}}$$

Problem: Aşağıda denge konumunda verilen sistemin diferansiyel denklemini çıkarıp tabii frekansını hesaplayınız



Çözüm:



Newton'un 2. kanunu uygulanırsa,

$$\Sigma M = J_{Top}\ddot{\varphi}$$

$$J_m\ddot{\varphi} + M\,\ddot{x}.L = -k\,x.L$$

$\varphi << 0 \Rightarrow \sin\varphi \cong \varphi \quad \cos\varphi \cong 1$ alınabilir.

Yaydaki sıkışma miktarı $x = L.\sin\varphi = L\,\varphi$

$$\dot{x} = L\,\dot{\varphi}$$

$$\ddot{x} = L\,\ddot{\varphi}$$

$$J_m = \frac{1}{3} m\,L^2$$

Yukarıdaki denklemi düzenleyelim.

$$\left(\frac{1}{3}m\,L^2 + M\,L^2\right)\ddot{\varphi} + k\,L^2\,\varphi = 0$$

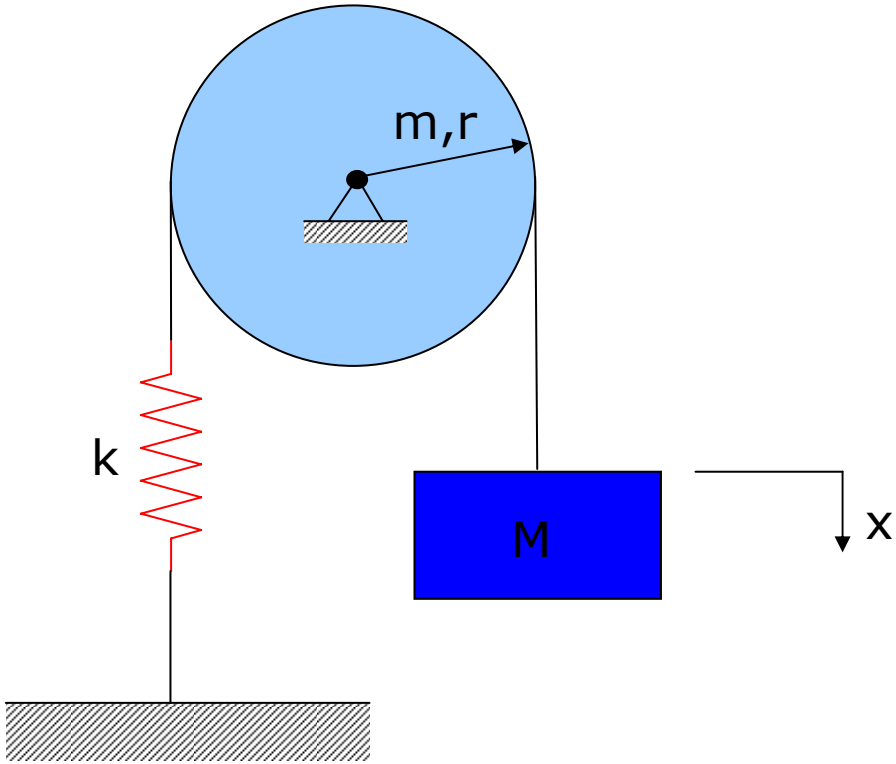
sadeleştirme yapılırsa,

$$\left(\frac{1}{3}m + M\right)\ddot{\varphi} + k\,\varphi = 0$$

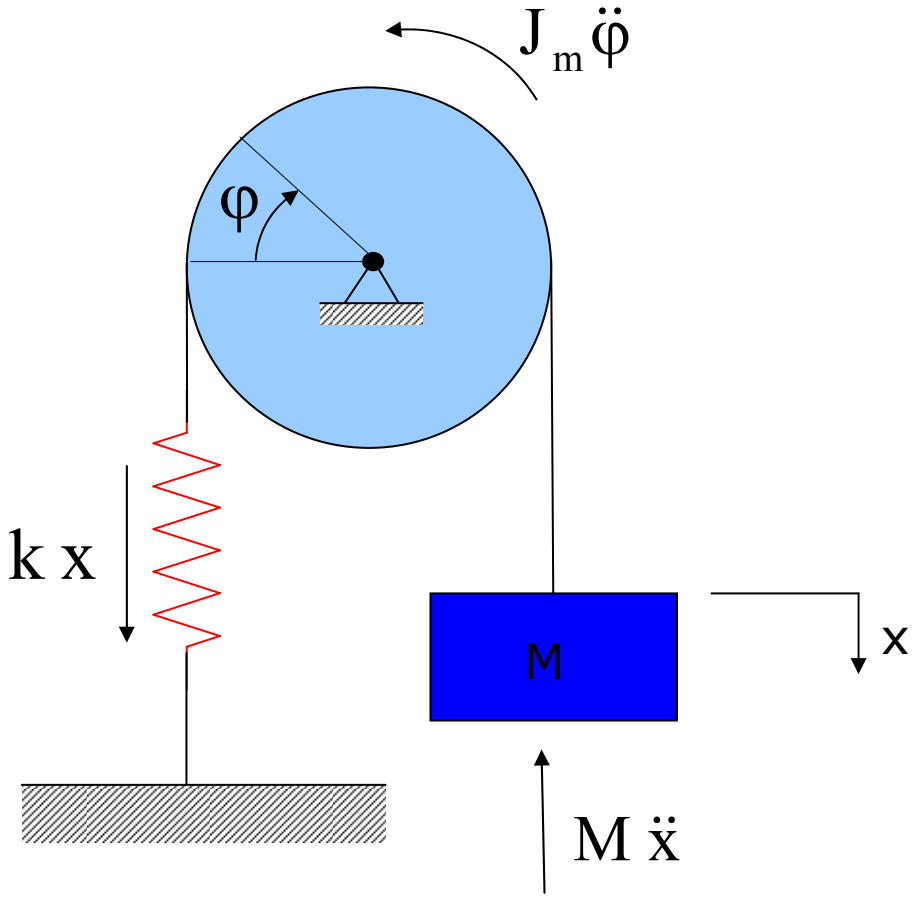
bulunur.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{\frac{1}{3}m + M}} = \sqrt{\frac{3\,k}{m + 3\,M}} \quad \text{rad/s}$$

Problem: Aşağıda denge konumunda verilen sistemin diferansiyel denklemini çıkarıp tabii frekansını hesaplayınız



Çözüm:



Newton'un 2. kanunu uygulanırsa,

$$\Sigma M = J_{Top}\ddot{\varphi}$$

$$J_m\ddot{\varphi} + M\,\ddot{x}.r = -k\,x.r \qquad J_m = \frac{1}{2}m\,r^2$$

$$\varphi << 0 \;\Rightarrow\; \sin\varphi \cong \varphi \quad \cos\varphi \cong 1$$

$$x = r.\sin\varphi = r\,\varphi$$

$$\dot{x} = r\,\dot{\varphi}$$

$$\ddot{x} = r\,\ddot{\varphi}$$

yazılabilir.

düzenleme yapılırsa,

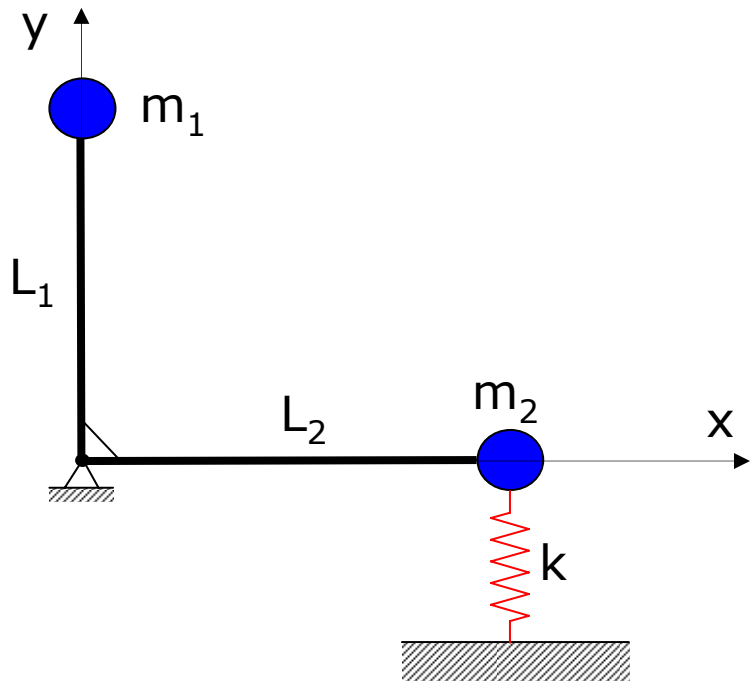
$$\left(\frac{1}{2}\,m\,r^2\right)\ddot{\varphi} + M\,\ddot{x}.r = -k\,x.r$$

$$\left(\frac{1}{2}\,m\,r^2 + M\,r^2\right)\ddot{\varphi} + k\,r^2\varphi = 0$$

$$\left(\frac{1}{2}\,m + M\right)\ddot{\varphi} + k\,\varphi = 0$$

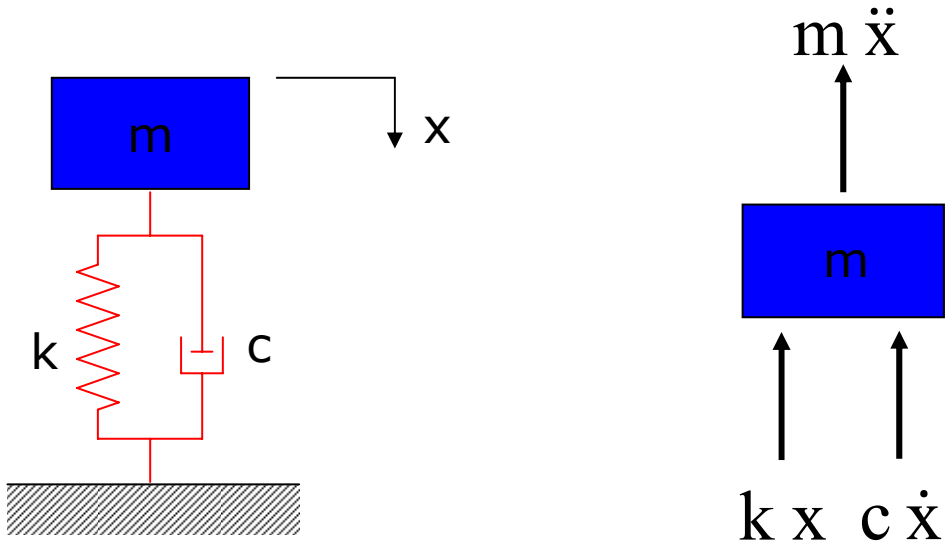
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{\frac{1}{2}\,m + M}} = \sqrt{\frac{2\,k}{m + 2\,M}} \qquad \text{rad/s}$$

Problem: Aşağıda denge konumunda verilen sistemin diferansiyel denklemini çıkarıp tabii frekansını hesaplayınız



Sönümlü Serbest Titreşim

Hareketin diferansiyel denkleminin bulunması:



Newton'un 2. kanununa göre,

$$\Sigma F = m\,a \quad \Rightarrow \quad m\,\ddot{x} = -c\,\dot{x} - k\,x \quad \Rightarrow \quad m\,\ddot{x} + c\,\dot{x} + k\,x = 0$$

Sönümlü Serbest Titreşim Hareket Denkleminin Bulunması

$$m\ddot{x} + c\dot{x} + kx = 0$$

Bu diferansiyel denklemin çözümünün

$$x = A\,e^{st}$$

biçiminde olduğunu biliyoruz. Burada, A ve s integrasyon sabitleridir. Çözüm kabulünü türetirsek,

$$\dot{x} = s\,A\,e^{st}$$

$$\ddot{x} = s^2 A\,e^{st}$$

Bulunur. Bunlar yukarıdaki diferansiyel denklemde yerine konurda,

$$\left(m\,s^2 + c\,s + k\right) A\,e^{st} = 0$$

Burada, $A\,,\ e^{s\,t} \neq 0$ dır.

$$m\,s^2 + c\,s + k = 0$$

Bulunur, bu denkleme **karakteristik denklem** denir.
Karakteristik denklemin kökleri,

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

dir.

Sistemin birbirinden bağımsız iki gerçek kökü vardır. Bu durumda, hareket denklemi:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

A_1 ve A_2 başlangıç şartlarından bulunacak katsayılardır.

Kritik Sönüm Katsayısı ve Sönüm Oranı

Kritik sönüm katsayısı c_{kr} aşağıdaki gibi tanımlanır.

$$\left(\frac{c_{kr}}{2m}\right)^2 - \frac{k}{m} = 0$$

$$c_{kr} = 2m\sqrt{\frac{k}{m}} = 2\sqrt{k\,m} = 2m\omega_n$$

Sönüm oranı ise,

$$\xi = \frac{c}{c_{kr}}$$ olarak tanımlanır.

Eğer karakteristik denklem,

$$m\,s^2 + c\,s + k = 0$$

ξ ve ω_n cinsinden yazılırsa,

$$s^2 + \underbrace{\frac{c}{m}}_{2\xi\omega_n}\,s + \underbrace{\frac{k}{m}}_{\omega_n^2} = 0 \quad \Rightarrow \quad s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

denklemin kökleri aşağıdaki gibi bulunur.

$$x(t) = A_1\, e^{\left(-\xi+\sqrt{\xi^2-1}\right)\omega_n t} + A_2\, e^{\left(-\xi-\sqrt{\xi^2-1}\right)\omega_n t}$$

1.Durum: Zayıf Sönümlü Sistem

$$\xi < 1, \quad c_{kr} > c, \quad \frac{c}{2m} < \sqrt{\frac{k}{m}}$$

Kritik altı sönümlü sistemlerde $\left(\xi^2 - 1\right)$ negatif olur. Bu durumda,

$$s_1 = \left(-\xi + i\sqrt{1-\xi^2}\right)\omega_n$$
$$s_2 = \left(-\xi - i\sqrt{1-\xi^2}\right)\omega_n$$

olur. Sistemin bir çift eşlenik kompleks kökü olduğundan,

$$x(t) = A_1 e^{\left(-\xi + i\sqrt{1-\xi^2}\right)\omega_n t} + A_2 e^{\left(-\xi - i\sqrt{1-\xi^2}\right)\omega_n t}$$

$$x(t) = e^{-\xi\,\omega_n t}\left(A_1 e^{i\sqrt{1-\xi^2}\,\omega_n t} + A_2 e^{-i\sqrt{1-\xi^2}\,\omega_n t}\right)$$

$$x(t) = e^{-\xi \omega_n t}\left[A_1\left(\cos\sqrt{1-\xi^2}\,\omega_n t + i\sin\sqrt{1-\xi^2}\,\omega_n t\right) + A_2\left(\cos\sqrt{1-\xi^2}\,\omega_n t - i\sin\sqrt{1-\xi^2}\,\omega_n t\right)\right]$$

$$x(t) = e^{-\xi \omega_n t}\left[\underbrace{\left(A_1 + A_2\right)}_{B_1}\cos\sqrt{1-\xi^2}\,\omega_n t + i\underbrace{\left(A_1 - A_2\right)}_{B_2}\sin\sqrt{1-\xi^2}\,\omega_n t\right]$$

$$x(t) = X\, e^{-\xi \omega_n t}\sin\left(\sqrt{1-\xi^2}\,\omega_n t + \phi\right)$$

$$x(t) = X\, e^{-\xi \omega_n t}\cos\left(\sqrt{1-\xi^2}\,\omega_n t - \phi\right)$$

$$\left.\begin{aligned} X &= \sqrt{B_1^2 + B_2^2} \\ \phi &= \tan^{-1}\frac{B_2}{B_1} \end{aligned}\right\}$$ olarak yazılabilir.

Başlangıç şartları, $\begin{cases} x\big|_{t=0} = x_0 \\ \dot{x}\big|_{t=0} = \dot{x}_0 \end{cases}$ ise,

$$B_1 = x_0$$

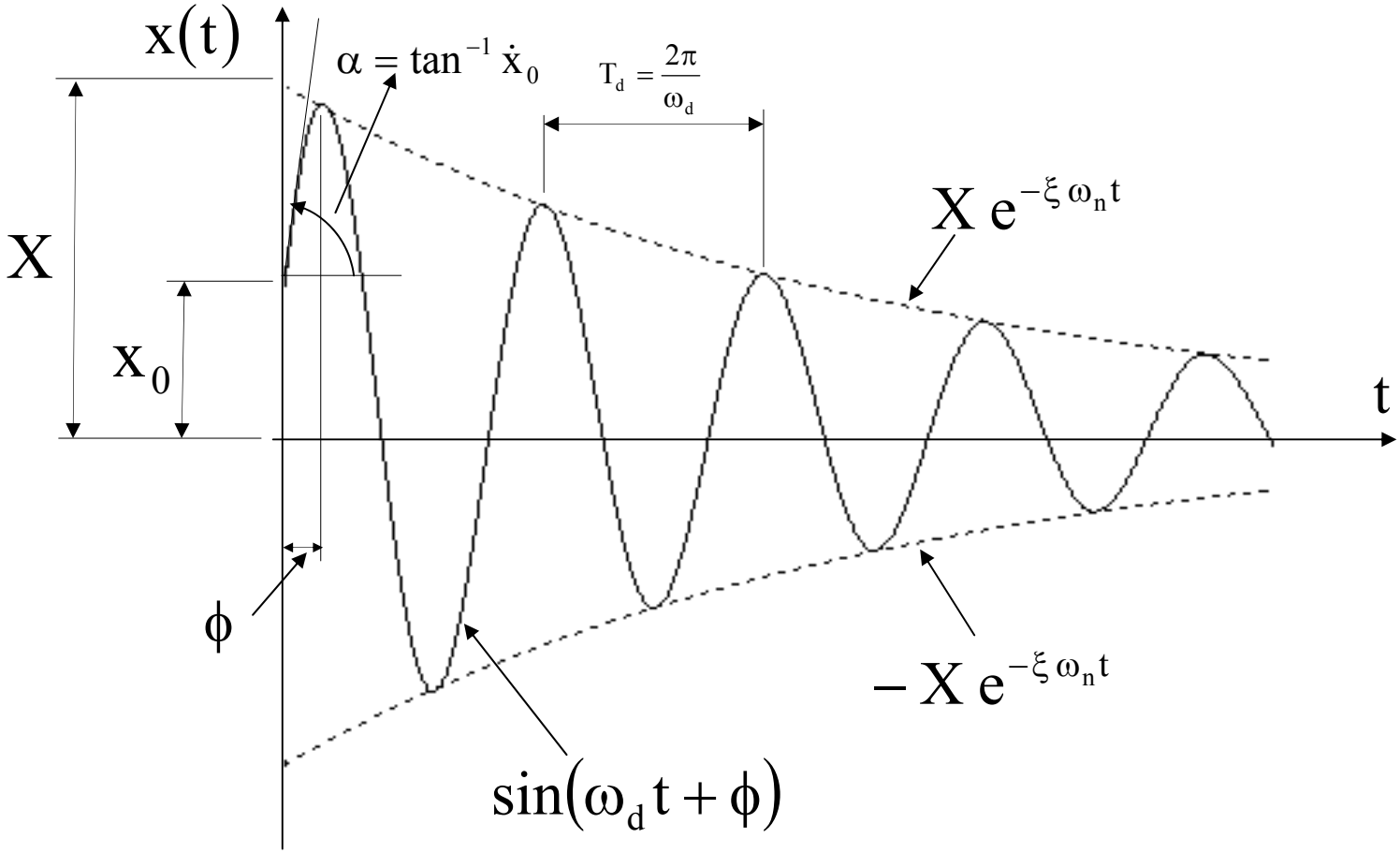
$$B_2 = \frac{\dot{x}_0 + \xi\,\omega_n x_0}{\omega_n\sqrt{1-\xi^2}}$$ olarak bulunur. Bu durumda, $x(t)$

$$x(t) = e^{-\xi \omega_n t}\left\{ x_0 \cos\left(\sqrt{1-\xi^2}\,\omega_n t\right) + \frac{\dot{x}_0 + \xi \omega_n x_0}{\omega_n \sqrt{1-\xi^2}} \sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)\right\}$$

$\omega_d = \sqrt{1-\xi^2}\,\omega_n$ dir.

$$x(t) = e^{-\xi \omega_n t}\left\{ x_0 \cos(\omega_d t) + \frac{\dot{x}_0 + \xi \omega_n x_0}{\omega_n \sqrt{1-\xi^2}} \sin(\omega_d t)\right\}$$

olarak bulunur.


x(t)
$\alpha = \tan^{-1} \dot{x}_0$
$T_d = \frac{2\pi}{\omega_d}$
X
x_0
$X e^{-\xi \omega_n t}$
t
ϕ
$-X e^{-\xi \omega_n t}$
$\sin(\omega_d t + \phi)$

2.Durum: Kritik Sönümlü Sistem

$$\xi = 1, \quad c_{kr} = c, \quad \frac{c}{2m} = \sqrt{\frac{k}{m}}$$

Katlı kök olduğundan, $s_1 = s_2 = -\frac{c_{kr}}{2m} = -\omega_n$

$m\ddot{x} + c\dot{x} + kx = 0$ denklemin çözümü:

$x(t) = (A_1 + A_2 t)e^{-\omega_n t}$ formunda olacaktır.

Başlangıç şartları, $\begin{cases} x|_{t=0} = x_0 \\ \dot{x}|_{t=0} = \dot{x}_0 \end{cases}$ ise,

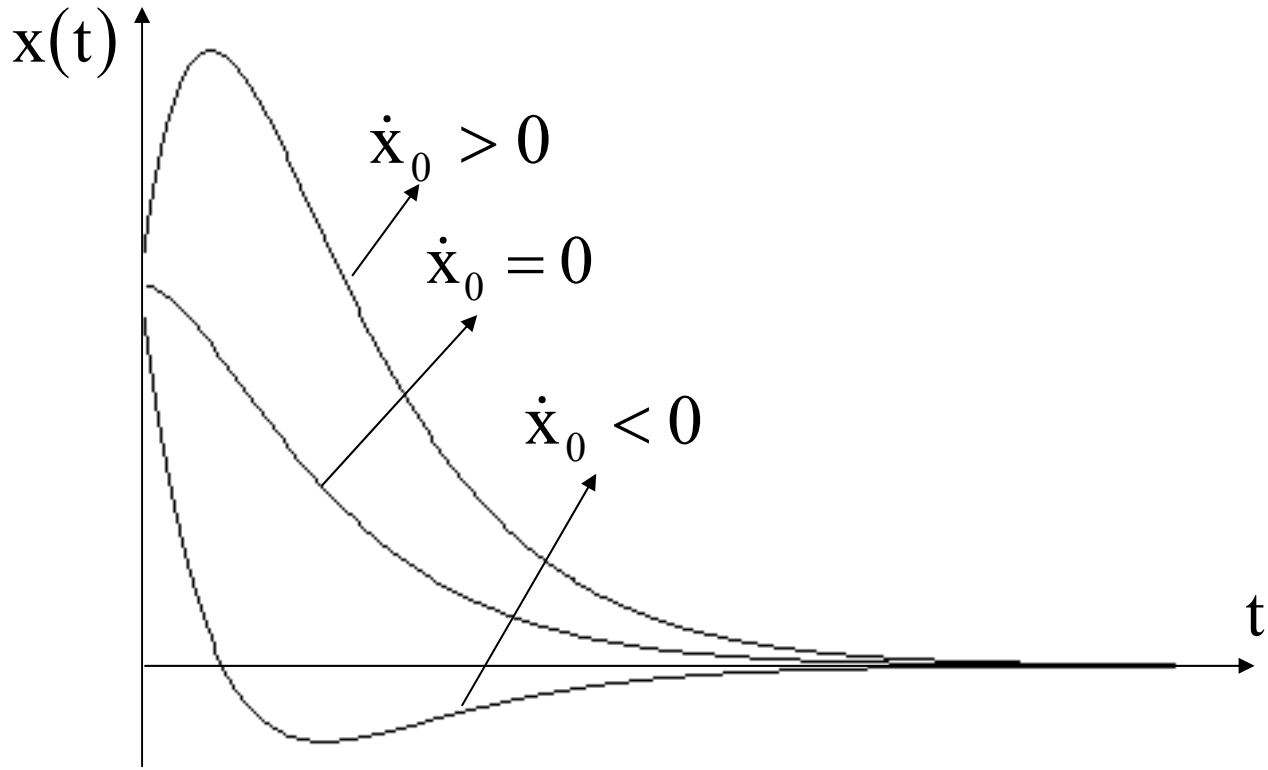
$A_1 = x_0$

olarak bulunur ve çözüm:

$A_2 = \dot{x}_0 + \omega_n x_0$

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0\right)\right] e^{-\omega_n t}$$

bulunur. Dikkat edilirse, $t \to \infty \Rightarrow e^{-\omega_n t} \to 0$



3.Durum: Aşırı Sönümlü Sistem

$$\xi > 1, \quad c > c_{kr}, \quad \frac{c}{2m} > \sqrt{\frac{k}{m}}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

Diferansiyel denkleminin karakteristik denklemi,

$$ms^2 + cs + k = 0$$ 'in kökleri:

$$s_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n$$ dir.

eğer, $\xi > 1 \Rightarrow \sqrt{\xi^2 - 1} > 0$ olur.

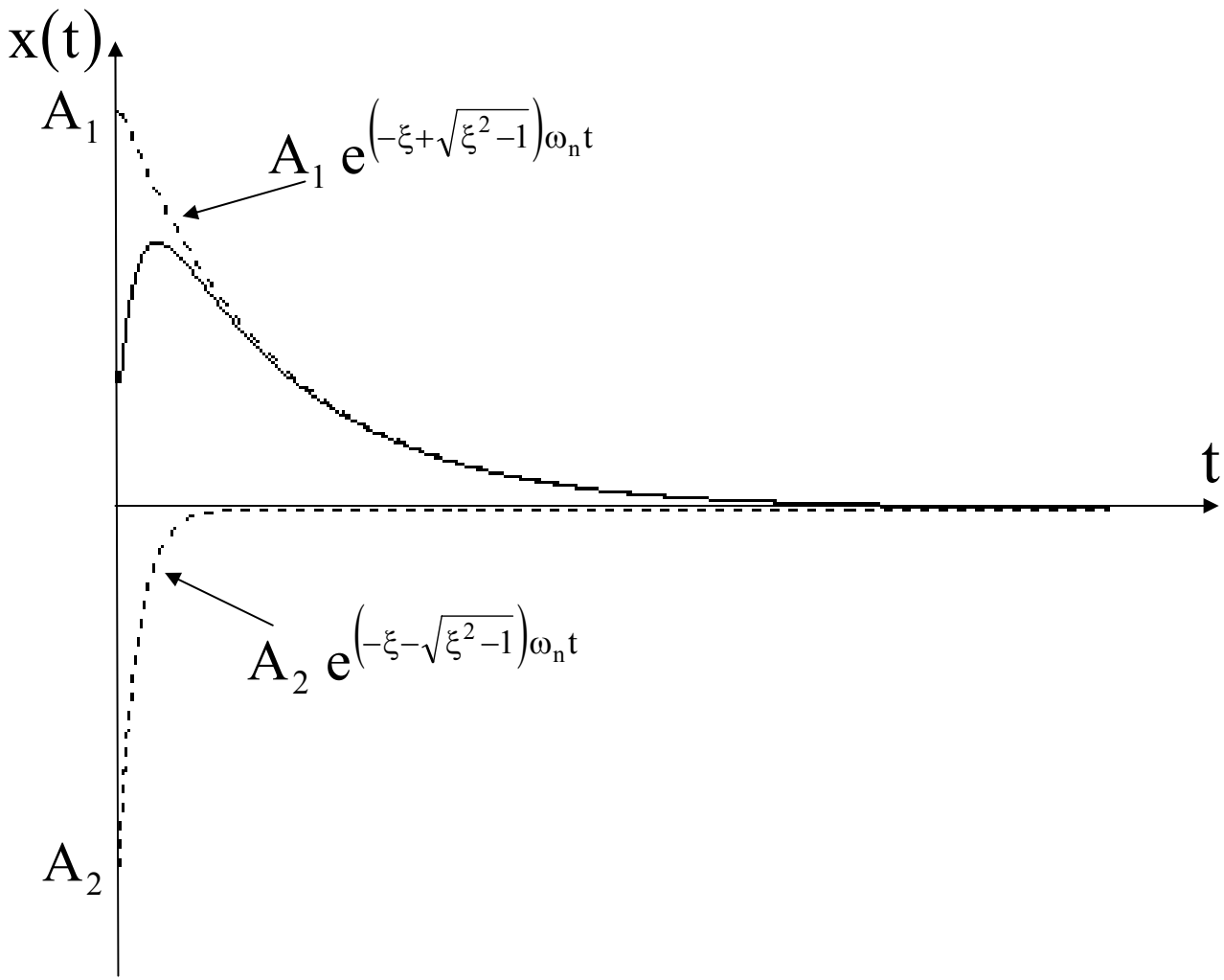
Bu durumda, kökler reel ve ayrık olacaktır. Denklemin çözümü:

$$x(t) = A_1 e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} + A_2 e^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}$$

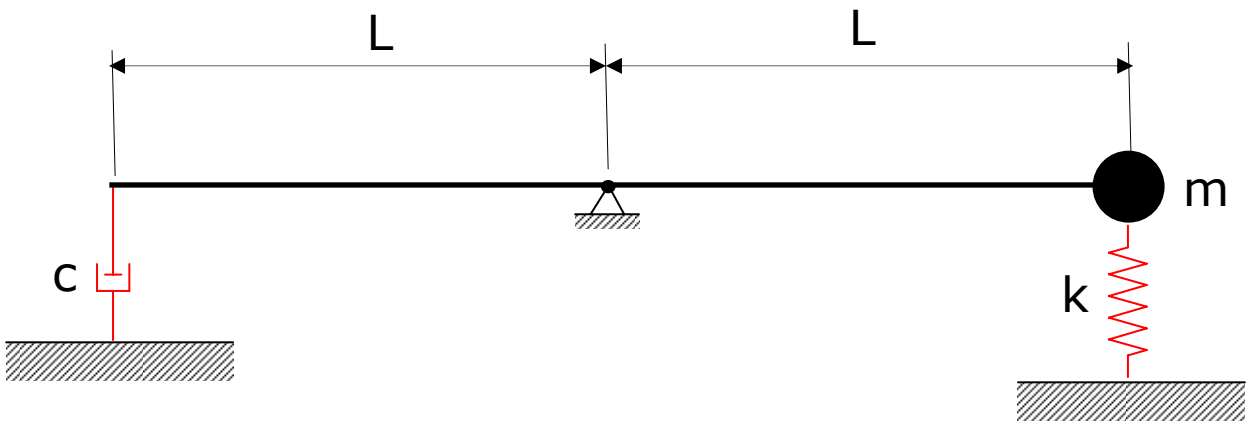
Başlangıç şartları, $\begin{cases} x\big|_{t=0} = x_0 \\ \dot{x}\big|_{t=0} = \dot{x}_0 \end{cases}$ ise,

$$A_1 = \frac{x_0\omega_n\left(\xi + \sqrt{\xi^2 - 1}\right) + \dot{x}_0}{2\omega_n\sqrt{\xi^2 - 1}}$$

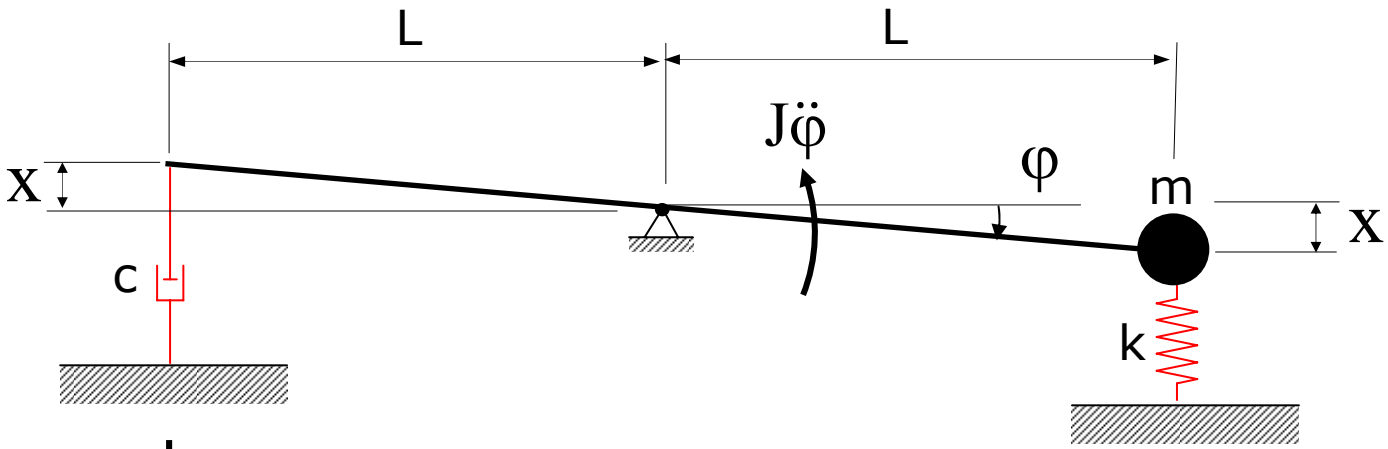
$$A_2 = \frac{-x_0\omega_n\left(\xi - \sqrt{\xi^2 - 1}\right) - \dot{x}_0}{2\omega_n\sqrt{\xi^2 - 1}}$$


x(t)
A1
A1 e(−ξ+√(ξ²−1))ωnt
t
A2 e(−ξ−√(ξ²−1))ωnt
A2

Problem: Aşağıda denge konumundaki sistemin verilen değer ve başlangıç şartlarına bağlı olarak 5 s için hareketini inceleyiniz.



Çözüm:



$$x = L\sin\varphi \cong L\varphi$$

$$\dot{x} = L\dot{\varphi}$$

$$\ddot{x} = L\ddot{\varphi}$$

$$J = mL^2$$

$c\,\dot{x}$

$k\,x$

$$m = 50\,kg \quad k = 15\,10^3\ N/m \quad c = 25\ N.s/m \quad L = 1.2\,m$$

$$t = 0 \begin{cases} x_0 = \varphi_0 = \dfrac{3\pi}{180}\,rad \\ \dot{x}_0 = \dot{\varphi}_0 = \dfrac{5\pi}{180}\,rad/s \end{cases}$$

Newton'un 2. kanunu uygulanırsa,

$$\sum M = J_{Top}\ddot{\varphi}$$

$$J\ddot{\varphi} = -c\dot{x}\,L - kx\,L$$

$$mL^2\ddot{\varphi} + cL^2\dot{\varphi} + kL^2\varphi = 0 \quad \Rightarrow \quad 72\ddot{\varphi} + 36\dot{\varphi} + 21600\varphi = 0$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{5.5127\pi} = 0.362798\,\mathrm{s} \quad f_d = \frac{1}{T_d} = 2.25635\,\mathrm{Hz}$$

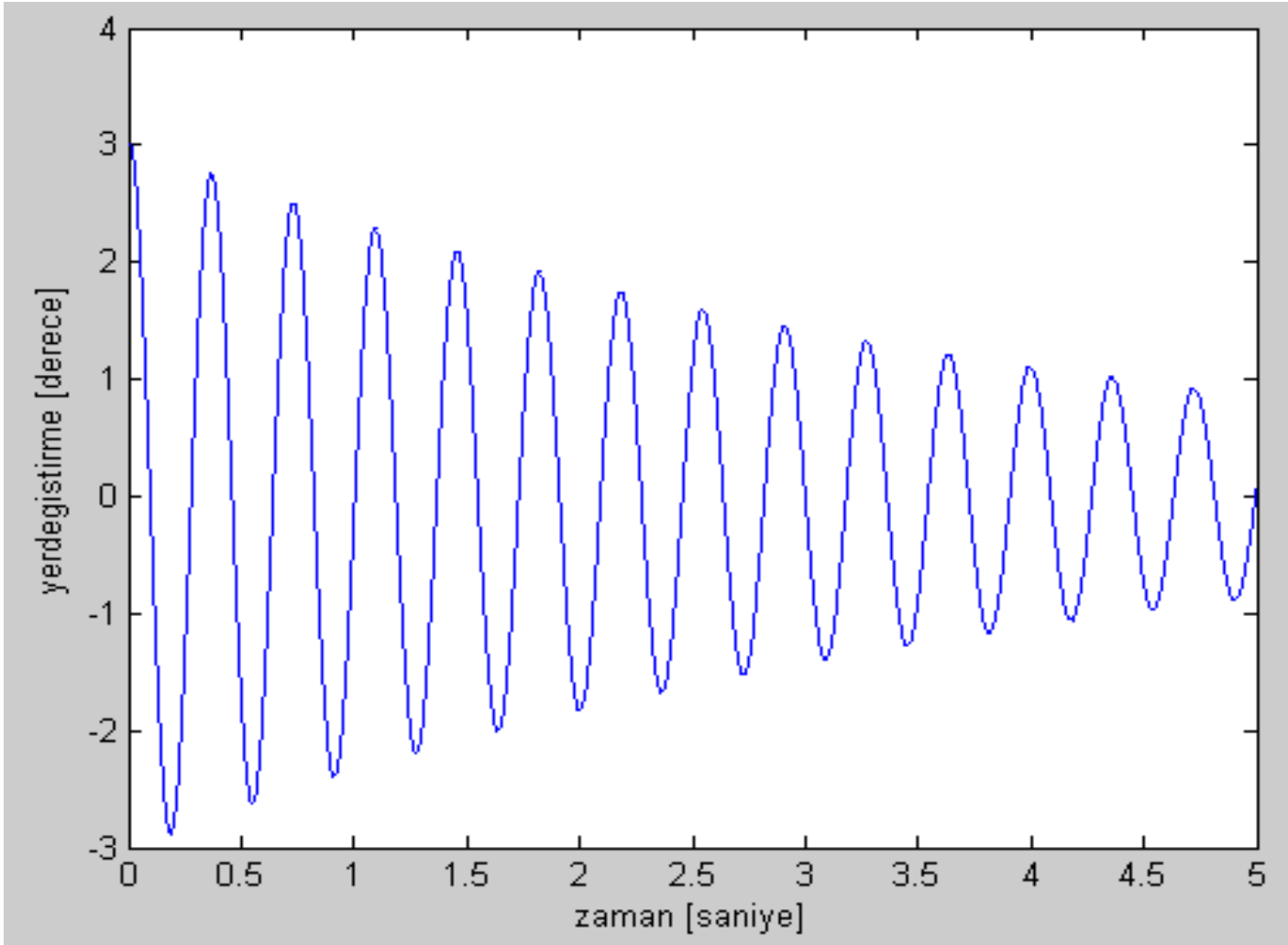
$$\xi = \frac{c}{c_{kr}} = \frac{c}{2\sqrt{k\,m}} = \frac{36}{2\sqrt{21600\ \ 72}} = 0.0144$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{21600}{72}} = 5.5132\pi\ \mathrm{rad/s}$$

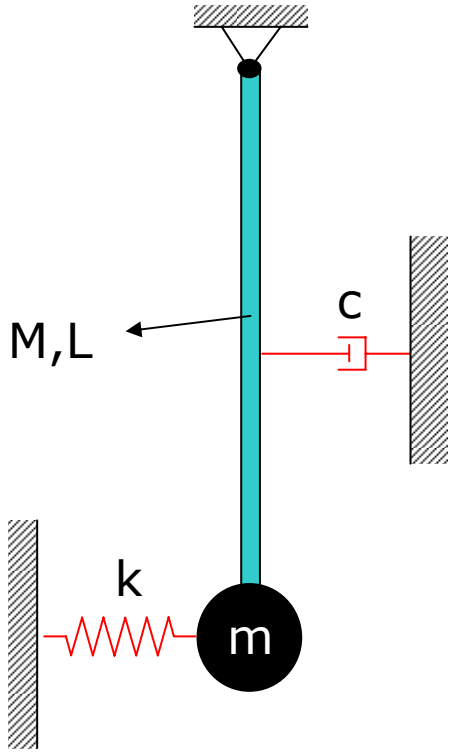
$$\omega_d = \omega_n\sqrt{1-\xi^2} = 5.5132\pi\sqrt{1-0.0144^2} = 5.5127\pi\ \mathrm{rad/s}$$

$$\varphi(t) = e^{-\xi\,\omega_n t}\left\{\varphi_0\cos(\omega_d t) + \frac{\dot{\varphi}_0 + \xi\,\omega_n\varphi_0}{\omega_n\sqrt{1-\xi^2}}\sin(\omega_d t)\right\}$$

$$\varphi(t) = e^{-0.2494\,t}\left[0.0523598\ \ \cos(17.3186\,t) + 0.0057929\ \ \sin(17.3186\,t)\right]$$


yerdegistirme [derece]
zaman [saniye]
4
3
2
1
0
-1
-2
-3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Problem: Aşağıda denge konumundaki sistemin verilen değer ve başlangıç şartlarına bağlı olarak 5 s için hareketini inceleyiniz.

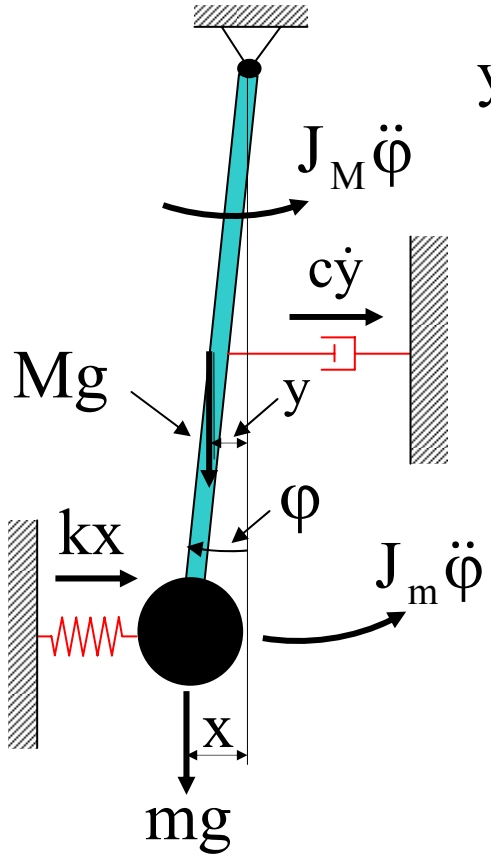


Çözüm:

$$x = L\varphi \quad \dot{x} = L\dot{\varphi} \quad \ddot{x} = L\ddot{\varphi}$$

$$y = \frac{L}{2}\varphi \quad \dot{y} = \frac{L}{2}\dot{\varphi} \quad \ddot{y} = \frac{L}{2}\ddot{\varphi}$$

$$J = \frac{1}{3}ML^2$$



Newton’un 2. kanunu uygulanırsa,

$$\sum M = J\ddot{\varphi}$$

$$J_m\ddot{\varphi} + J_M\ddot{\varphi} = -c\dot{y}\frac{L}{2} - kx\,L - mg\,x - Mg\,y$$

$$\left(mL^2 + \frac{1}{3}ML^2\right)\ddot{\varphi} + c\frac{L^2}{4}\dot{\varphi} + \left(kL^2 + mgL + Mg\frac{L}{2}\right)\varphi = 0$$

$$12\ddot{\varphi} + 5\dot{\varphi} + 12000\varphi = 0$$

$$m = 10\,\text{kg} \quad M = 6\,\text{kg} \quad L = 1\,\text{m} \quad c = 5\,\text{Ns/m} \quad k = 12\,10^3\ \text{N/m}$$

$$t = 0 \begin{cases} x_0 = \varphi_0 = \dfrac{10\pi}{180}\,\text{rad} \\ \dot{x}_0 = \dot{\varphi}_0 = \dfrac{5\pi}{180}\,\text{rad/s} \end{cases}$$

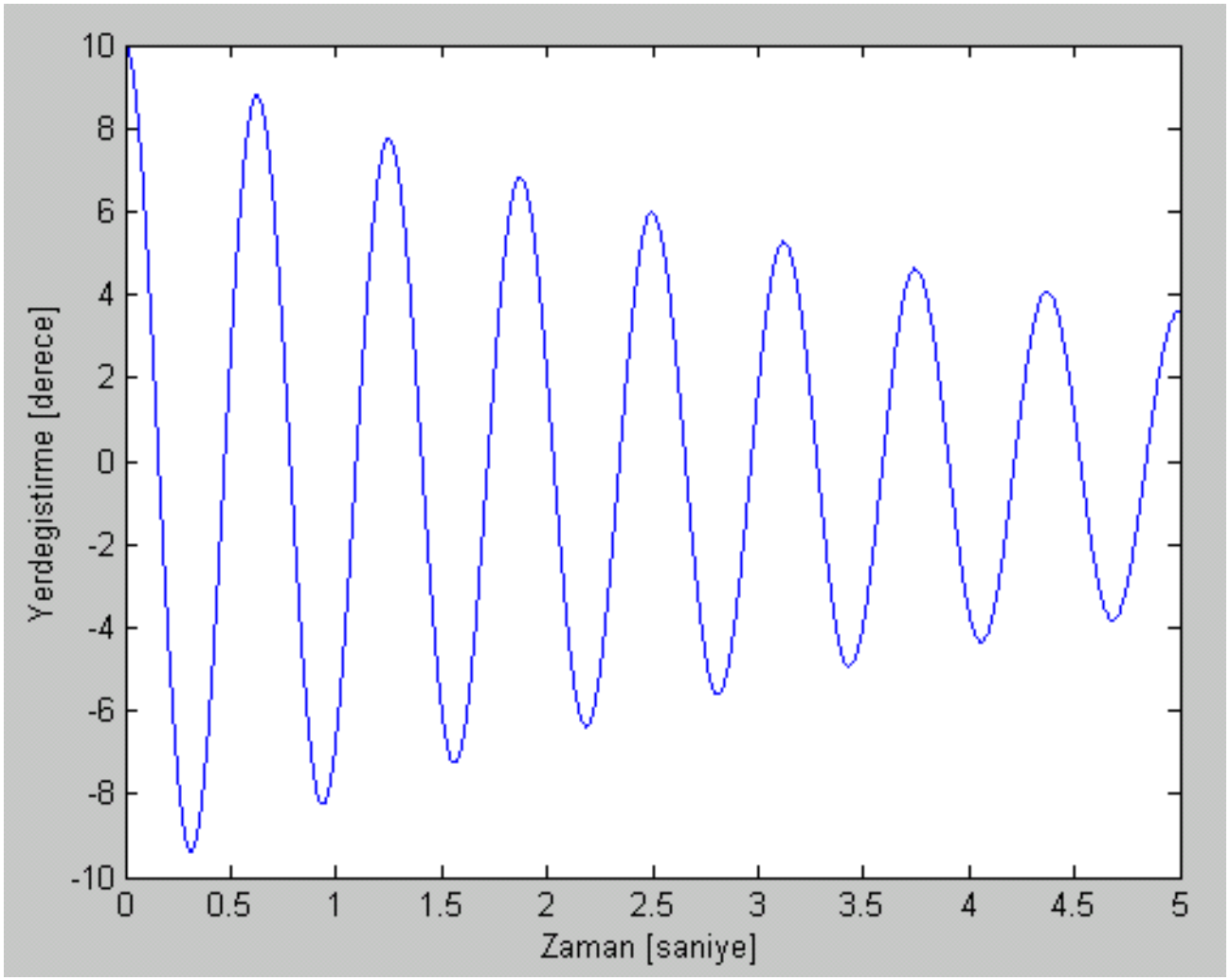
$$\xi = \frac{c}{c_{kr}} = \frac{c}{2\sqrt{km}} = \frac{5}{2\sqrt{12000\ 12}} = 0.0065$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12000}{12}} = 10.065\pi \text{ rad/s}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 10.065\pi\sqrt{1-0.0065^2} \cong 10.065\pi$$

$$\varphi(t) = e^{-\xi\omega_n t}\left\{\varphi_0\cos(\omega_d t) + \frac{\dot{\varphi}_0 + \xi\omega_n\varphi_0}{\omega_n\sqrt{1-\xi^2}}\sin(\omega_d t)\right\}$$

$$\varphi(t) = e^{-0.2055t}\left\{0.1745\cos(10.065\pi t) + 0.00389\sin(10.065\pi t)\right\}$$


Yerdegistirme [derece]
Zaman [saniye]
10
8
6
4
2
0
-2
-4
-6
-8
-10
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5